

Dynamic Epistemic Temporal Logic

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Abstract

We introduce a new type of arrow in the *update frames* (or “action models”) of *Dynamic Epistemic Logic* in a way that enables us to reason about epistemic temporal dynamics in multi-agent systems that *need not be synchronous*. Since van Benthem and Pacuit (later joined by Hoshi and Gerbrandy) showed that standard Dynamic Epistemic Logic necessarily satisfies *synchronicity*, it follows that our arrow type is a new way of extending the domain of applicability of the Dynamic Epistemic Logic approach. Furthermore, our framework provides a new perspective on the van Benthem et al work itself. In particular, while each of our work and their work shows that epistemic temporal models generated by standard update frames necessarily satisfy certain structural properties such as synchronicity, our work clarifies the way in which these structural properties arise as a result of the inherent structure of standard update frames themselves. In the conclusion of the paper, we point toward forthcoming work in which we show that the basic idea behind our new update-frame arrow-type stems from a much more general extension of Dynamic Epistemic Logic in which formulas are used to specify *arrow preconditions* in an update frame, enabling us to describe the condition under which an arrow is to be inserted in the Kripke model produced by the operation on Kripke models that the update frame describes. This novel generalization brings a much broader range of multi-agent systems within the reach of the Dynamic Epistemic Logic approach, thereby presenting a potential new direction for research in this area.

1 Introduction

Dynamic Epistemic Logic [2, 3, 4, 10, 17, 19] is a modal-logic approach to reasoning about belief dynamics in multi-agent systems. The characteristic feature of this approach is its use of *update modals*, which are modal operators $[U, s]$ that describe operations on Kripke models. These operations, called *updates*, represent *informational events* in which the agents receive information that may bring about changes in their beliefs. The basic idea is that an update modal $[U, s]$ describes a specific partial function $f_{[U, s]}$ that maps a pointed Kripke

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32 model (M, w) in the domain of $f_{[U,s]}$ to another pointed Kripke model that we write as
 33 $(M[U], (w, s))$. This allows us to view a sequence

$$(M_0, w_0), (M_1, w_1), (M_2, w_2), \dots, (M_n, w_n) \quad (1)$$

34 of pointed Kripke models, with (M_{i+1}, w_{i+1}) generated from (M_i, w_i) by the update $f_{[U_{i+1}, s_{i+1}]}$
 35 described by update modal $[U_{i+1}, s_{i+1}]$, as a discrete-time distributed multi-agent system in
 36 which the state of the system at time i is described by (M_i, w_i) . Defining the *time of a world*
 37 w in M_i within the sequence (1) to be the index i , we obtain a notion of time that is *external*
 38 to the pointed Kripke model (M_i, w_i) . One consequence of adopting this external notion of
 39 time is that all of the worlds that an agent considers possible relative to a world w in M_i
 40 have time i . This implies that at every world, every agent knows the current time. Systems
 41 in which the current time is known at every world are called *synchronous* [15, 16]. Dynamic
 42 Epistemic Logic, which itself adopts this external notion of time, is consequently restricted
 43 to the study of *synchronous* multi-agent systems [15, 16].

44 In this paper, we propose a simple extension to the update modals $[U, s]$ that allows us to
 45 reason about discrete-time distributed multi-agent systems that *need not be synchronous*. We
 46 achieve this by adapting the methodology of standard Dynamic Epistemic Logic so that it fits
 47 naturally within a version of *Epistemic Temporal Logic* [6, 9] whose only temporal modality
 48 is a discrete one-step-past operator; this version will be called *Simple Epistemic Temporal*
 49 *Logic*. Simple Epistemic Temporal Logic uses *epistemic temporal models*, which are Kripke
 50 models in which one of the relational components is designated as a *time-keeping relation*.
 51 When w is related to w' according to the time-keeping relation, the intended interpretation is
 52 that w' is a possible way the system might have been one time-step before w . This provides us
 53 with an *internal* notion of time, in that the *time of a world w in an epistemic temporal model*
 54 M is determined solely based on the time-keeping relation, which is internal to the model
 55 M . Diagrammatically, we will represent this relation using arrows labeled by the symbol
 56 Y —called *Y-arrows*—where “ Y ” is a mnemonic for “yesterday” (so having a Y -arrow from
 57 world w to world w' is to be thought of as saying that w' is one of the possible ways w might
 58 have been “yesterday,” meaning one time-step ago). In order to distinguish between Kripke
 59 models with and without a *Y-relation* (the time-keeping relation), we adopt the following
 60 terminology: *epistemic temporal models* are Kripke models with a designated Y -relation—
 61 these have an *internal* notion of time—whereas *epistemic models* are Kripke models without
 62 a designated Y -relation—these have an *external* notion of time. Since an epistemic temporal
 63 model M uses an *internal* notion of time, the ways in which the system described by M can
 64 evolve are determined in advance by the structure of the Y -relation in M ; said informally,
 65 the *protocol is fixed*. In contrast, the protocol in Dynamic Epistemic Logic is *dynamic*, as
 66 it can be changed on-the-fly by using a different update modal to produce the next pointed
 67 Kripke model appearing in the sequence (1).

68 In extending the updates of standard Dynamic Epistemic Logic from the class of epistemic
 69 models (having *external* time) to the class of epistemic temporal models (having *internal*
 70 time), we stand to gain *dynamic protocols* for systems that *need not be synchronous*. While
 71 standard Dynamic Epistemic Logic sets each world in $M[U]$ to be one time-step ahead of
 72 any world in M , our new updates on epistemic temporal models allow us greater flexibility
 73 in modeling the passage of time. In particular, using the internal notion of time associated
 74 with the Y -relation, our updates allow us to let worlds in $M[U]$ have any natural-number

75 time; therefore, in certain updates that embed M into $M[U]$, each world in $M[U]$ can be
76 seen either as a world in M or else as an arbitrarily distant possible future of a world in
77 M . Such flexibility is essential to the study of *asynchronous* systems. To bring about this
78 flexibility, we add a new structural component to update modals: the \underline{Y} -arrow. We use
79 \underline{Y} -arrows to specify exact positions in which the update $f_{[U,s]}$ is to insert Y -arrows in the
80 updated model $M[U]$. We then identify sufficient conditions on our new update modals $[U, s]$
81 that will guarantee that the update $f_{[U,s]}$ preserves properties such as synchronicity in the
82 resulting epistemic temporal model. We use these conditions to show that epistemic temporal
83 models that result from sequentially applying our new kinds of updates are *isomorphic* to the
84 generated sequences of epistemic models from standard Dynamic Epistemic Logic that have
85 been studied by a number of authors [8, 12, 15, 16, 20]. While [15, 16] showed that properties
86 such as synchronicity are necessary of standardly generated sequences, our isomorphism
87 result pinpoints the necessity of these properties within the structure of standard update
88 modals $[U, s]$ themselves. This provides a *new perspective* on the results of [15, 16].

89 In the next section, we introduce the language L_{DETL} and the theory T_{DETL} of *Dynamic*
90 *Epistemic Temporal Logic*. It is this theory that we use in reasoning about our new kinds
91 of updates on epistemic temporal models. Due to space constraints, we will omit the proofs
92 of our results; the interested reader can find full details in [11], an extended version of this
93 paper.

94 2 Syntax

95 Our theory T_{DETL} and its language L_{DETL} concern the beliefs of a nonempty finite set A of
96 agents, none of whose names coincide with the special symbols Y or \underline{Y} .

97 **Notation 2.1** (A, Y, \underline{Y}). A is a finite nonempty set of symbols not containing the symbols
98 Y and \underline{Y} . The members of A will be called *agents*.

99 **Definition 2.2.** Let S be a nonempty set of symbols. A *Kripke frame* (for S) is a tuple
100 $F = (W, R)$ for which

- 101 • W is a nonempty set whose elements will be called *worlds* (in F) and
- 102 • $R : S \rightarrow (W \rightarrow 2^W)$ is a function mapping each symbol $a \in S$ to a function $R_a : W \rightarrow$
103 2^W that maps each world $w \in W$ to a set of worlds $R_a(w) \subseteq W$.¹

104 To say that a Kripke frame $F = (W, R)$ is *finite* means that W is finite. Notation: for a
105 Kripke frame F , we write W^F to denote the first element of the tuple F and we write R^F
106 to denote the second element of the tuple F .

107 **Notation 2.3** (A, Y, \underline{Y}). A is a finite nonempty set of symbols not containing the symbols
108 Y and \underline{Y} . The members of A will be called *agents*. The symbol Y is called *yesterday*, and
109 the symbol \underline{Y} is called *yesterday-intro*.

¹The function R_a^M gives rise to a binary relation $\bar{R}_a^M := \{(x, y) \in W^M \times W^M : y \in R_a^M(x)\}$ on W^M . We will conflate R_a^M and \bar{R}_a^M whenever it is convenient. We will often refer to the members of \bar{R}_a^M as *a-arrows*.

110 **Definition 2.4.** For a language L having formulas, an L -update frame is a tuple $U =$
 111 (W, R, \mathbf{p}) for which

- 112 • (W, R) is a finite Kripke frame for $A \cup \{Y, \underline{Y}\}$ that will be called the Kripke frame
 113 *underlying* U , and
- 114 • $\mathbf{p} : W \rightarrow L$ is a function mapping each world $s \in W$ to an L -formula $\mathbf{p}(s)$.

115 A *state* in U is just a world in the Kripke frame underlying U . Notation: for an L -update
 116 frame U , we write W^U to denote the first element of the tuple U , we write R^U to denote
 117 the second element of the tuple U , and we write \mathbf{p}^U to denote the third element of the tuple
 118 U . A *pointed L -update frame* is a pair (U, s) consisting of an L -update frame U and a world
 119 $s \in W^U$ that will be called the *point* of (U, s) .

120 Update frames are also called “action models” (or “event models”) in the Dynamic Epis-
 121 temic Logic literature [2, 3, 4, 10, 17, 19]. For an update frame U , a state $s \in W^U$ represents
 122 the communication of the formula $\mathbf{p}^U(s)$. For an agent $a \in A$, the relation R_a^U represents
 123 agent a ’s conditional uncertainty as to which formula is communicated: if $s' \in R_a^U(s)$ and
 124 the formula $\mathbf{p}^U(s)$ was in fact communicated, then agent a will think that the formula $\mathbf{p}^U(s')$
 125 is one of the formulas that might have been communicated.

126 We now define our language L_{DETL} as an extension of the language L_{ETL} of Simple Epis-
 127 temic Temporal Logic.

128 **Definition 2.5** (L_{ETL}). The L_{ETL} , the *Language of (Simple) Epistemic Temporal Logic*,
 129 consists of the formulas formed by the following grammar.

$$\begin{aligned} \varphi ::= & \perp \mid \top \mid p_k \mid \varphi \star \varphi \mid \neg\varphi \mid [a]\varphi \\ & k \in \mathbb{N}, \star \in \{\rightarrow, \vee, \wedge, \equiv\}, a \in A \cup \{Y\} \end{aligned}$$

130 Terminology: for each $k \in \mathbb{N}$, we call p_k a *propositional letter*; for each $a \in A$, we call $[a]$ a
 131 *doxastic modal*; we call $[Y]$ the *yesterday modal*. For each agent $a \in A$, we read the formula
 132 $[a]\varphi$ as “agent a believes that φ is true.” We read the formula $[Y]\varphi$ as “ φ is true in all
 133 possible yesterdays.” Notation: for each $a \in A \cup \{Y\}$, we let $\langle a \rangle$ abbreviate $\neg[a]\neg$ and we let
 134 $\langle U, s \rangle$ abbreviate $\neg[U, s]\neg$; we define for each $i \in \mathbb{N}$ the formula $[a]^i\varphi$ by setting $[a]^0\varphi := \varphi$
 135 and $[a]^{i+1}\varphi := [a]([a]^i\varphi)$; the formula $\langle a \rangle^i$ is defined analogously.

136 **Definition 2.6** (L_{DETL}). L_{DETL} is the *Language of Dynamic Epistemic Temporal Logic*. The
 137 L_{DETL} -formulas are the formulas that may be formed by the grammar obtained from that
 138 in Definition 2.5 by adding the following formula-formation rule: if φ is an L_{DETL} -formula
 139 and (U, s) is a pointed L -update frame with $\emptyset \neq L \subseteq L_{\text{DETL}}$, then $[U, s]\varphi$ is an L_{DETL} -
 140 formula. L_{DETL} consists of the L_{DETL} -formulas along with the L -update frames for which
 141 $\emptyset \neq L \subseteq L_{\text{DETL}}$. Terminology: we call $[U, s]$ an *update modal*; we let $\langle U, s \rangle$ abbreviate
 142 $\neg[U, s]\neg$. We read the formula $[U, s]\varphi$ as “after update (U, s) , φ is true.”

143 **Definition 2.7.** An *update frame* is an L_{DETL} -update frame. A *formula* is a L_{DETL} -formula.

144 **Definition 2.8.** T_{DETL} , the *Theory of Asynchronous Dynamic Epistemic Logic*, is defined in
 145 Figure 1. For a formula φ , we write $\vdash \varphi$ to mean that φ is a T_{DETL} -theorem and we write
 146 $\not\vdash \varphi$ to mean that φ is not a T_{DETL} -theorem.

BASIC SCHEMES

CL.	Schemes for Classical Propositional Logic
K_a .	$[a](\varphi \rightarrow \psi) \rightarrow ([a]\varphi \rightarrow [a]\psi)$ for $a \in A$
K_Y .	$[Y](\varphi \rightarrow \psi) \rightarrow ([Y]\varphi \rightarrow [Y]\psi)$
UA.	$[U, s]q \equiv (\mathbf{p}^U(s) \rightarrow q)$ for $q \in \{p_k, \perp, \top\}$
$U\star$.	$[U, s](\varphi \star \psi) \equiv ([U, s]\varphi \star [U, s]\psi)$ for $\star \in \{\rightarrow, \vee, \wedge, \equiv\}$
$U\neg$.	$[U, s]\neg\varphi \equiv (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)$
$U[a]$.	$[U, s][a]\varphi \equiv (\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_a^U(s)} [a][U, s']\varphi)$ for $a \in A$
$U[Y]$.	$[U, s][Y]\varphi \equiv (\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y][U, s']\varphi) \wedge$ $(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_{\perp}^U(s)} [U, s']\varphi)$

RULES

$$\frac{\vdash \varphi \rightarrow \psi \quad \vdash \varphi}{\vdash \psi} \text{(MP)} \quad \frac{a \in A \cup \{Y\} \quad \vdash \varphi}{\vdash [a]\varphi} \text{(MN)} \quad \frac{\vdash \varphi}{\vdash [U, s]\varphi} \text{(UN)}$$

Figure 1: The theory T_{DETL}

147 Since our interest here is in implementing update mechanisms on Kripke models with a
 148 designated Y -relation, we do not impose any of the usual properties on belief or on time
 149 that one might expect [6, 8, 12, 15, 16, 20]. So T_{DETL} should be viewed as the *minimal*
 150 theory that brings update mechanisms to Simple Epistemic Temporal Logic. Future work
 151 will investigate extensions of this theory that include familiar restrictions on belief and on
 152 time, though we do address the preservation of certain time-related properties in Section 4.

153 **Definition 2.9** (Adapted from [19]). We define a function $c : L_{\text{DETL}} \rightarrow \mathbb{N}$ by the following
 154 induction on formula construction, with $|W^U|$ denoting the (nonzero, finite) number of states
 155 in U .

$$\begin{aligned} c(q) &:= 1 \text{ for } q \in \{p_k, \perp, \top\} \\ c(\varphi \star \psi) &:= 1 + \max\{c(\varphi), c(\psi)\} \\ c(\neg\varphi) &:= 1 + c(\varphi) \\ c([a]\varphi) &:= 1 + c(\varphi) \text{ for } a \in A \cup \{Y\} \\ c([U, s]\varphi) &:= (4 + c(U)) \cdot c(\varphi) \\ c(U) &:= (1 + |W^U|) \cdot \max_{s \in W^U} c(\mathbf{p}^U(s)) \end{aligned}$$

156 The *complexity* of a formula φ is the natural number $c(\varphi)$.

157 **Theorem 2.10.** The function $c : L_{\text{DETL}} \rightarrow \mathbb{N}$ from Definition 2.9 satisfies each of the
 158 following.

- 159 1. If ψ is a proper subformula of φ , then $c(\psi) < c(\varphi)$.

- 160 2. $c(\mathbf{p}^U(s)) < c(U)$.
- 161 3. $c(\mathbf{p}^U(s)) < c([U, s']\varphi)$.
- 162 4. $c(\mathbf{p}^U(s) \rightarrow q) < c([U, s]q)$ for $q \in \{p_k, \perp, \top\}$.
- 163 5. $c([U, s]\varphi \star [U, s]\psi) < c([U, s](\varphi \star \psi))$.
- 164 6. $c(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi) < c([U, s]\neg\varphi)$.
- 165 7. $c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) < c([U, s][a]\varphi)$ for $a \in A \cup \{Y\}$.
- 166 8. $c(\mathbf{p}^U(s) \rightarrow [U, s']\varphi) < c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi)$ for $a \in A \cup \{Y\}$.
- 167 9. If $c(\varphi) < c(\psi)$, then $c([U, s]\varphi) < c([U, s]\psi)$.

168 *Proof.* We prove each item in turn.

- 169 1. If ψ is a subformula of φ , then $c(\psi) < c(\varphi)$.

170 By inspection of Definition 2.9.

- 171 2. $c(\mathbf{p}^U(s)) < c(U)$.

$$\begin{aligned}
c(\mathbf{p}^U(s)) &\leq \max_{s' \in W^U} c(\mathbf{p}^U(s')) \\
&< (1 + |W^U|) \cdot \max_{s' \in W^U} c(\mathbf{p}^U(s')) \\
&= c(U)
\end{aligned}$$

- 172 3. $c(\mathbf{p}^U(s)) < c([U, s']\varphi)$.

$$\begin{aligned}
c(\mathbf{p}^U(s)) &< c(U) \\
&< (4 + c(U)) \cdot c(\varphi) \\
&= c([U, s']\varphi)
\end{aligned}$$

- 173 4. $c(\mathbf{p}^U(s) \rightarrow q) < c([U, s]q)$ for $q \in \{p_k, \perp, \top\}$.

$$\begin{aligned}
c(\mathbf{p}^U(s) \rightarrow q) &= 1 + \max\{c(\mathbf{p}^U(s)), c(q)\} \\
&= 1 + c(\mathbf{p}^U(s)) \\
&< 4 + c(U) \\
&= c([U, s]q)
\end{aligned}$$

- 174 5. $c([U, s]\varphi \star [U, s]\psi) < c([U, s](\varphi \star \psi))$.

$$\begin{aligned}
&c([U, s]\varphi \star [U, s]\psi) \\
&= 1 + \max\{(4 + c(U)) \cdot c(\varphi), (4 + c(U)) \cdot c(\psi)\} \\
&= 1 + (4 + c(U)) \cdot \max\{c(\varphi), c(\psi)\} \\
&< (4 + c(U)) \cdot (1 + \max\{c(\varphi), c(\psi)\}) \\
&= c([U, s](\varphi \star \psi))
\end{aligned}$$

$$175 \quad 6. \quad c(p^U(s) \rightarrow \neg[U, s]\varphi) < c([U, s]\neg\varphi).$$

$$\begin{aligned} & c(p^U(s) \rightarrow \neg[U, s]\varphi) \\ &= 1 + \max\{c(p^U(s)), 1 + (4 + c(U)) \cdot c(\varphi)\} \\ &= 2 + (4 + c(U)) \cdot c(\varphi) \\ &< (4 + c(U)) \cdot (1 + c(\varphi)) \\ &= c([U, s]\neg\varphi) \end{aligned}$$

$$176 \quad 7. \quad c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) < c([U, s][a]\varphi) \text{ for } a \in A \cup \{Y\}.$$

$$\begin{aligned} & c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) \\ &= 1 + \max\{c(\mathbf{p}^U(s)), 1 + (4 + c(U)) \cdot c(\varphi)\} \\ &= 2 + (4 + c(U)) \cdot c(\varphi) \\ &< (4 + c(U)) \cdot (1 + c(\varphi)) \\ &= c([U, s][a]\varphi) \end{aligned}$$

$$177 \quad 8. \quad c(\mathbf{p}^U(s) \rightarrow [U, s']\varphi) < c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) \text{ for } a \in A \cup \{Y\}.$$

$$\begin{aligned} & c(\mathbf{p}^U(s) \rightarrow [U, s']\varphi) \\ &= 1 + \max\{c(\mathbf{p}^U(s)), (4 + c(U)) \cdot c(\varphi)\} \\ &= 1 + (4 + c(U)) \cdot c(\varphi) \\ &< 1 + (4 + c(U)) \cdot (1 + c(\varphi)) \\ &= 1 + \max\{c(\mathbf{p}^U(s)), (4 + c(U)) \cdot (1 + c(\varphi))\} \\ &= c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) \end{aligned}$$

$$178 \quad 9. \quad \text{If } c(\varphi) < c(\psi), \text{ then } c([U, s]\varphi) < c([U, s]\psi).$$

$$\begin{aligned} c([U, s]\varphi) &= (4 + c(U)) \cdot c(\varphi) \\ &< (4 + c(U)) \cdot c(\psi) \\ &= c([U, s]\psi) \end{aligned}$$

179

□

180 **Theorem 2.11** (Reduction Theorem). By an induction on the complexity of formulas, we
 181 simultaneously define a function $\circ : L_{\text{DETL}} \rightarrow L_{\text{ETL}}$ according to the cases in Figure 2 and
 182 show that for each formula φ , we have that $\vdash \varphi \equiv \varphi^\circ$ and that $c(\varphi^\circ) \leq c(\varphi)$, with the
 183 inequality strict in case φ has the form $[U, u]\psi$.

184 *Proof.* By induction on $n \in \mathbb{N}$, we argue that the following four items hold for each formula
 185 φ with $c(\varphi) = n$: (i) the line in Figure 2 defining φ° is legitimate, by which we mean that
 186 on the line in Figure 2 in which “ φ° ” appears to the left of “ $:=$ ”, each expression of the
 187 form χ° appearing on this line to the right of “ $:=$ ” satisfies $c(\chi) < c(\varphi)$; (ii) $\varphi^\circ \in L_{\text{ETL}}$; (iii)
 188 $\vdash \varphi \equiv \varphi^\circ$; and (iv) $c(\varphi^\circ) \leq c(\varphi)$, with the inequality strict in case φ has the form $[U, u]\psi$.

$$\begin{aligned}
q^\circ &:= q \text{ for } q \in \{p_k, \perp, \top\} \\
(\varphi \star \psi)^\circ &:= \varphi^\circ \star \psi^\circ \\
(\neg\varphi)^\circ &:= \neg(\varphi^\circ) \\
([a]\varphi)^\circ &:= [a](\varphi^\circ) \text{ for } a \in A \\
([Y]\varphi)^\circ &:= [Y](\varphi^\circ) \\
([U, s]q)^\circ &:= (\mathbf{p}^U(s))^\circ \rightarrow q \text{ for } q \in \{p_k, \perp, \top\} \\
([U, s](\varphi \star \psi))^\circ &:= ([U, s]\varphi)^\circ \star ([U, s]\psi)^\circ \\
([U, s]\neg\varphi)^\circ &:= (\mathbf{p}^U(s))^\circ \rightarrow \neg([U, s]\varphi)^\circ \\
([U, s][a]\varphi)^\circ &:= (\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_a^U(s)} [a]([U, s']\varphi)^\circ \text{ for } a \in A \\
([U, s][Y]\varphi)^\circ &:= ((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y]([U, s']\varphi)^\circ) \wedge \\
&\quad ((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_{\underline{Y}}^U(s)} ([U, s']\varphi)^\circ) \\
([U, s][U', s']\varphi)^\circ &:= ([U, s]([U', s']\varphi)^\circ)^\circ
\end{aligned}$$

Figure 2: Definition of $\circ : L_{\text{DETL}} \rightarrow L_{\text{ETL}}$

189 Many of the cases of the induction are handled similarly, the argument depending only on
190 the last two steps in the construction of a given formula according to the grammar for L_{DETL}
191 (Definition 2.6). We consider each of the possible cases in turn. Note that we sometimes use
192 “IH” as an abbreviation for “induction hypothesis.”

- 193 • For $q \in \{p_k, \perp, \top\}$: $\vdash q \equiv q^\circ$ and $c(q^\circ) = c(q)$.

194 We have $q^\circ = q$ by the definition of \circ . (i) follows immediately, (ii) follows by the
195 definition of L_{ETL} , (iii) follows by propositional reasoning, and (iv) follows immediately.

- 196 • $\vdash (\varphi \star \psi) \equiv (\varphi \star \psi)^\circ$ and $c((\varphi \star \psi)^\circ) \leq c(\varphi \star \psi)$.

197 By Theorem 2.10 (1), we have that $c(\varphi) < c(\varphi \star \psi)$ and $c(\psi) < c(\varphi \star \psi)$. So (i) follows,
198 and we may apply the induction hypothesis to each of φ and ψ .

199 For (ii), we apply the induction hypothesis to each of φ and ψ , from which it follows
200 that $\varphi^\circ \in L_{\text{ETL}}$ and $\psi^\circ \in L_{\text{ETL}}$. We therefore have that $(\varphi^\circ \star \psi^\circ) \in L_{\text{ETL}}$ by the
201 definition of L_{ETL} , and thus that $(\varphi \star \psi)^\circ \in L_{\text{ETL}}$ by the definition of \circ .

202 For (iii), we apply the induction hypothesis to each of φ and ψ , from which it follows
203 that $\vdash \varphi \equiv \varphi^\circ$ and $\vdash \psi \equiv \psi^\circ$. Applying propositional reasoning, we have that
204 $\vdash (\varphi \star \psi) \equiv (\varphi^\circ \star \psi^\circ)$. Since we have $(\varphi^\circ \star \psi^\circ) = (\varphi \star \psi)^\circ$ by the definition of \circ , (iii)
205 follows.

206 For (iv), we apply the induction hypothesis to each of φ and ψ , from which it follows
207 that $c(\varphi^\circ) \leq c(\varphi)$ and $c(\psi^\circ) \leq c(\psi)$, with the respective inequalities strict in case
208 φ or ψ has the form $[U, u]\chi$. Applying the definition of c , we have $c(\varphi^\circ \star \psi^\circ) =$
209 $1 + \max\{c(\varphi^\circ), c(\psi^\circ)\}$ and $c(\varphi \star \psi) = 1 + \max\{c(\varphi), c(\psi)\}$, so (iv) follows.

- 210 • $\vdash \neg\varphi \equiv (\neg\varphi)^\circ$ and $c((\neg\varphi)^\circ) \leq c(\neg\varphi)$.

211 By Theorem 2.10 (1), we have that $c(\varphi) < c(\neg\varphi)$. So (i) follows, and we may apply
 212 the induction hypothesis to φ .

213 For (ii), we apply the induction hypothesis to φ , from which it follows that $\varphi^\circ \in L_{\text{ETL}}$.
 214 We therefore have that $\neg\varphi^\circ \in L_{\text{ETL}}$ by the definition of L_{ETL} , and thus that $(\neg\varphi)^\circ \in$
 215 L_{ETL} by the definition of \circ .

216 For (iii), we apply the induction hypothesis to φ , from which it follows that $\vdash \varphi \equiv \varphi^\circ$.
 217 Applying propositional reasoning, we have that $\vdash \neg\varphi \equiv \neg(\varphi^\circ)$. Since we have $\neg(\varphi^\circ) =$
 218 $(\neg\varphi)^\circ$ by the definition of \circ , (iii) follows.

219 For (iv), we apply the induction hypothesis to φ , from which it follows that $c(\varphi^\circ) \leq$
 220 $c(\varphi)$, with the inequality strict in case φ has the form $[U, u]\psi$. Applying the definition
 221 of c , we have $c(\neg(\varphi^\circ)) = 1 + c(\varphi^\circ)$ and $c(\neg\varphi) = 1 + c(\varphi)$, so (iv) follows.

222 • For $a \in A \cup \{Y\}$: $\vdash [a]\varphi \equiv ([a]\varphi)^\circ$ and $c(([a]\varphi)^\circ) \leq c([a]\varphi)$.

223 By Theorem 2.10 (1), $c(\varphi) < c([a]\varphi)$. So (i) follows, and we may apply the induction
 224 hypothesis to φ .

225 For (ii), we apply the induction hypothesis to φ , from which it follows that $\varphi^\circ \in L_{\text{ETL}}$.
 226 We therefore have that $[a]\varphi^\circ \in L_{\text{ETL}}$ by the definition of L_{ETL} , and thus that $([a]\varphi)^\circ \in$
 227 L_{ETL} by the definition of \circ .

228 For (iii), we apply the induction hypothesis to φ , from which it follows that $\vdash \varphi \equiv \varphi^\circ$.
 229 Applying modal reasoning, we have that $\vdash [a]\varphi \equiv [a](\varphi^\circ)$. Since we have $[a](\varphi^\circ) =$
 230 $([a]\varphi)^\circ$ by the definition of \circ , (iii) follows.

231 For (iv), we apply the induction hypothesis to φ , from which it follows that $c(\varphi^\circ) \leq$
 232 $c(\varphi)$, with the inequality strict in case φ has the form $[U, u]\psi$. Applying the definition
 233 of c , we have $c([a](\varphi^\circ)) = 1 + c(\varphi^\circ)$ and $c([a]\varphi) = 1 + c(\varphi)$, so (iv) follows.

234 • For $q \in \{p_k, \perp, \top\}$: $\vdash [U, s]q \equiv ([U, s]q)^\circ$ and $c(([U, s]q)^\circ) < c([U, s]q)$.

235 By Theorem 2.10 (4), we have $c(\mathbf{p}^U(s) \rightarrow q) < c([U, s]q)$. By Theorem 2.10 (1), we have
 236 $c(\mathbf{p}^U(s)) < (\mathbf{p}^U(s) \rightarrow q)$. So (i) follows, and we may apply the induction hypothesis to
 237 each of $\mathbf{p}^U(s)$ and $\mathbf{p}^U(s) \rightarrow q$.

238 For (ii), we apply the induction hypothesis to $\mathbf{p}^U(s) \rightarrow q$, from which it follows that
 239 $(\mathbf{p}^U(s) \rightarrow q)^\circ \in L_{\text{ETL}}$.

240 For (iii), we have $\vdash [U, s]q \equiv \mathbf{p}^U(s) \rightarrow q$ by Axiom UA. Applying the induction
 241 hypothesis to $\mathbf{p}^U(s) \rightarrow q$, it follows that $\vdash (\mathbf{p}^U(s) \rightarrow q) \equiv (\mathbf{p}^U(s) \rightarrow q)^\circ$ and thus
 242 that $\vdash (\mathbf{p}^U(s) \rightarrow q) \equiv ((\mathbf{p}^U(s))^\circ \rightarrow q)$ by the definition of \circ . Since we have $(\mathbf{p}^U(s))^\circ \rightarrow$
 243 $q = ([U, s]q)^\circ$ by the definition of \circ , (iii) follows.

We now prove (iv).

$$\begin{aligned}
& c([U, s]q)^\circ \\
= & c((\mathbf{p}^U(s))^\circ \rightarrow q) && \text{def. of } \circ \\
= & 1 + \max\{c((\mathbf{p}^U(s))^\circ), 1\} && \text{def. of } c \\
= & 1 + c((\mathbf{p}^U(s))^\circ) && c(\mathbf{p}^U(s)) \geq 1 \\
\leq & 1 + c(\mathbf{p}^U(s)) && \text{IH on } \mathbf{p}^U(s) \\
< & 4 + c(U) && \text{Thm. 2.10 (2)} \\
= & c([U, s]q) && \text{def. of } c; c(q) = 1
\end{aligned}$$

245

- $\vdash [U, s](\varphi \star \psi) \equiv ([U, s](\varphi \star \psi))^\circ$ and $c([U, s](\varphi \star \psi))^\circ < c([U, s](\varphi \star \psi))$.

246

By Theorem 2.10 (5), we have $c([U, s]\varphi \star [U, s]\psi) < c([U, s](\varphi \star \psi))$. By Theorem 2.10 (1), we have $c([U, s]\varphi) < c([U, s]\varphi \star [U, s]\psi)$ and $c([U, s]\varphi) < c([U, s]\varphi \star [U, s]\psi)$. So (i) follows, and we may apply the induction hypothesis to each of $[U, s]\varphi$, $[U, s]\psi$, and $[U, s]\varphi \star [U, s]\psi$.

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For (ii), we apply the induction hypothesis to $[U, s]\varphi \star [U, s]\psi$, from which it follows that $([U, s]\varphi \star [U, s]\psi)^\circ \in L_{\text{ETL}}$.

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For (iii), we have that $\vdash [U, s](\varphi \star \psi) \equiv ([U, s]\varphi \star [U, s]\psi)$ by Axiom $U\star$. Applying the induction hypothesis to $[U, s]\varphi \star [U, s]\psi$, it follows that $\vdash ([U, s]\varphi \star [U, s]\psi) \equiv ([U, s]\varphi \star [U, s]\psi)^\circ$ and thus that $\vdash ([U, s]\varphi \star [U, s]\psi) \equiv ([U, s]\varphi)^\circ \star ([U, s]\psi)^\circ$ by the definition of \circ . Since we have $([U, s]\varphi)^\circ \star ([U, s]\psi)^\circ = ([U, s](\varphi \star \psi))^\circ$ by the definition of \circ , (iii) follows.

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We now prove (iv).

$$\begin{aligned}
& c([U, s](\varphi \star \psi))^\circ \\
= & c([U, s]\varphi)^\circ \star ([U, s]\psi)^\circ && \text{def. of } \circ \\
= & 1 + \max\{c([U, s]\varphi)^\circ, c([U, s]\psi)^\circ\} && \text{def. of } c \\
\leq & 1 + \max\{c([U, s]\varphi), c([U, s]\psi)\} && \text{IH on } [U, s]\varphi, [U, s]\psi \\
= & c([U, s]\varphi \star [U, s]\psi) && \text{def. of } c \\
< & c([U, s](\varphi \star \psi)) && \text{Thm. 2.10 (5)}
\end{aligned}$$

258

- $\vdash [U, s]\neg\varphi \equiv ([U, s]\neg\varphi)^\circ$ and $c([U, s]\neg\varphi)^\circ < c([U, s]\neg\varphi)$.

259

By Theorem 2.10 (6), we have $c(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi) < c([U, s]\neg\varphi)$. By Theorem 2.10 (1), we have $c(\mathbf{p}^U(s)) < c(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)$ and $c([U, s]\varphi) < c(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)$. So (i) follows, and we may apply the induction hypothesis to each of $\mathbf{p}^U(s)$, $[U, s]\varphi$, and $\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi$.

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For (ii), we apply the induction hypothesis to $\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi$, from which it follows that $(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)^\circ \in L_{\text{ETL}}$.

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For (iii), we have that $\vdash [U, s]\neg\varphi \equiv (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)$ by Axiom $U\neg$. Applying the induction hypothesis to $\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi$, it follows that $\vdash (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi) \equiv (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)^\circ$ and thus that $\vdash (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi) \equiv ((\mathbf{p}^U(s))^\circ \rightarrow \neg([U, s]\varphi)^\circ)$ by the definition of \circ . Since we have that $((\mathbf{p}^U(s))^\circ \rightarrow \neg([U, s]\varphi)^\circ) = ([U, s]\neg\varphi)^\circ$ by the definition of \circ , (iii) follows.

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We now prove (iv).

$$\begin{aligned}
& c([U, s]\neg\varphi)^\circ \\
= & c((\mathbf{p}^U(s))^\circ \rightarrow \neg([U, s]\varphi)^\circ) && \text{def. of } \circ \\
= & 1 + \max\{c((\mathbf{p}^U(s))^\circ), 1 + c([U, s]\varphi)^\circ\} && \text{def. of } c \\
\leq & 1 + \max\{c(\mathbf{p}^U(s)), 1 + c([U, s]\varphi)\} && \text{IH on } \mathbf{p}^U(s), [U, s]\varphi \\
= & c(\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi) && \text{def. of } c \\
< & c([U, s]\neg\varphi) && \text{Thm. 2.10 (6)}
\end{aligned}$$

271

- For $a \in A$: $\vdash [U, s][a]\varphi \equiv ([U, s][a]\varphi)^\circ$ and $c([U, s][a]\varphi)^\circ < c([U, s][a]\varphi)$.

272

By Theorem 2.10 (7), we have $c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) < c([U, s][a]\varphi)$. By Theorem 2.10

273

(1), we have $c(\mathbf{p}^U(s)) < c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi)$ and $c([U, s']\varphi) < c(\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi)$.

274

So (i) follows, and we may apply the induction hypothesis to each of $\mathbf{p}^U(s)$, $[U, s']\varphi$,

275

and $\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi$.

276

For (ii), we apply the induction hypothesis to each of $\mathbf{p}^U(s)$ and $[U, s']\varphi$, from which

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it follows that $(\mathbf{p}^U(s))^\circ \in L_{\text{ETL}}$ and $([U, s']\varphi)^\circ \in L_{\text{ETL}}$. It then follows by the definition

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of L_{ETL} that $([U, s][a]\varphi)^\circ \in L_{\text{ETL}}$.

279

For (iii), we apply the induction hypothesis to $\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi$, from which it follows

280

that

$$\vdash (\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) \equiv (\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi)^\circ$$

281

for each $s' \in R_a^U(s)$. Applying the definition of \circ , we then have that

$$\vdash (\mathbf{p}^U(s) \rightarrow [a][U, s']\varphi) \equiv ((\mathbf{p}^U(s))^\circ \rightarrow [a]([U, s']\varphi)^\circ)$$

282

for each $s' \in R_a^U(s)$. Applying propositional reasoning, it follows that

$$\begin{aligned}
\vdash & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_a^U(s)} [a][U, s']\varphi \right) \equiv \\
& \left((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_a^U(s)} [a]([U, s']\varphi)^\circ \right) .
\end{aligned}$$

283

Applying Axiom U[a] and propositional reasoning, we then have that

$$\vdash [U, s][a]\varphi \equiv \left((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_a^U(s)} [a]([U, s']\varphi)^\circ \right) .$$

284

Since the right-hand side of the latter equivalence is equal to $([U, s][a]\varphi)^\circ$ by the defi-

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nition of \circ , (iii) follows.

286

We now prove (iv).

$$\begin{aligned}
& c([U, s][a]\varphi)^\circ \\
= & c((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_a^U(s)} [a]([U, s']\varphi)^\circ) && \text{def. of } \circ \\
\leq & 1 + \max\{(\mathbf{p}^U(s))^\circ, |W^U| + \max_{s' \in R_a^U(s)} c([U, s']\varphi)^\circ\} \\
\leq & 1 + \max\{\mathbf{p}^U(s), |W^U| + \max_{s' \in R_a^U(s)} c([U, s']\varphi)\} && \text{IH} \\
= & 1 + |W^U| + \max_{s' \in R_a^U(s)} c([U, s']\varphi) \\
= & 1 + |W^U| + \max_{s' \in R_a^U(s)} (4 + c(U)) \cdot c(\varphi) && \text{def. of } c \\
= & 1 + |W^U| + (4 + c(U)) \cdot c(\varphi) \\
< & (4 + c(U)) + (4 + c(U)) \cdot c(\varphi) && \text{Def. 2.9} \\
= & (4 + c(U)) \cdot (1 + c(\varphi)) \\
= & c([U, s][a]\varphi)
\end{aligned}$$

287

- $\vdash [U, s][Y]\varphi \equiv ([U, s][Y]\varphi)^\circ$ and $c(([U, s][Y]\varphi)^\circ) < c([U, s][Y]\varphi)$.

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By Theorem 2.10 (7), we have $c(\mathbf{p}^U \rightarrow [Y][U, s']\varphi) < c([U, s][Y]\varphi)$. By Theorem 2.10 (8), we have $c(\mathbf{p}^U \rightarrow [U, s']\varphi) < c(\mathbf{p}^U \rightarrow [Y][U, s']\varphi)$. By Theorem 2.10 (1), we have $c(\mathbf{p}^U(s)) < c(\mathbf{p}^U \rightarrow [U, s']\varphi)$ and $c([U, s']\varphi) < c(\mathbf{p}^U \rightarrow [U, s']\varphi)$. So (i) follows, and we may apply the induction hypothesis to each of $\mathbf{p}^U(s)$, $[U, s']\varphi$, $\mathbf{p}^U \rightarrow [U, s']\varphi$, and $\mathbf{p}^U \rightarrow [Y][U, s']\varphi$.

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For (ii), we apply the induction hypothesis to each of $\mathbf{p}^U(s)$ and $[U, s']\varphi$, from which it follows that $(\mathbf{p}^U(s))^\circ \in L_{\text{ETL}}$ and $([U, s']\varphi)^\circ \in L_{\text{ETL}}$. It then follows by the definition of L_{ETL} that $([U, s][Y]\varphi)^\circ \in L_{\text{ETL}}$.

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297

For (iii), similar to the case for $[U, s][a]\varphi$, one can argue (using Axiom $U[Y]$ instead of Axiom $U[a]$) that

$$\begin{aligned} \vdash [U, s][Y]\varphi &\equiv ((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y]([U, s']\varphi)^\circ) \wedge \\ &(\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} ([U, s']\varphi)^\circ \end{aligned}$$

298

299

But then we see that the right-hand side of the above equivalence is equal to $([U, s][Y]\varphi)^\circ$ by the definition of \circ , so (iii) follows.

300

We now prove (iv).

$$\begin{aligned} &c(([U, s][Y]\varphi)^\circ) \\ &= c((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y]([U, s']\varphi)^\circ \wedge \\ &\quad (\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} ([U, s']\varphi)^\circ) \\ &= 1 + c((\mathbf{p}^U(s))^\circ \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y]([U, s']\varphi)^\circ) \\ &\leq [\dots \text{ as in case for } [U, s][a]\varphi \dots] \\ &= 1 + (1 + |W^U| + (4 + c(U)) \cdot c(\varphi)) \\ &= 2 + |W^U| + (4 + c(U)) \cdot c(\varphi) \\ &< (4 + c(U)) \cdot (4 + c(U)) \cdot c(\varphi) \\ &= (4 + c(U)) \cdot (1 + c(\varphi)) \\ &= c([U, s][Y]\varphi) \end{aligned}$$

301

- $\vdash [U, s][U', s']\varphi \equiv ([U, s][U', s']\varphi)^\circ$ and $c(([U, s][U', s']\varphi)^\circ) < c([U, s][U', s']\varphi)$.

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By Theorem 2.10 (1), we have $c([U', s']\varphi) < c([U, s][U', s']\varphi)$. So we may apply the induction hypothesis to $[U', s']\varphi$ to conclude that $c(([U', s']\varphi)^\circ) < c([U', s']\varphi)$, noting that the inequality is strict due to the form of $[U', s']\varphi$. Using Theorem 2.10 (9), it then follows that $c([U, s]([U', s']\varphi)^\circ) < c([U, s][U', s']\varphi)$, from which we conclude that (i) holds. We also see that we may apply the induction hypothesis to each of $[U', s']\varphi$ and $[U, s]([U', s']\varphi)^\circ$.

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309

For (ii), we apply the induction hypothesis to $[U, s]([U', s']\varphi)^\circ$, from which it follows that $([U, s]([U', s']\varphi)^\circ)^\circ \in L_{\text{ETL}}$.

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312

For (iii), we apply the induction hypothesis to $[U', s']\varphi$, from which it follows that $\vdash [U', s']\varphi \equiv ([U', s']\varphi)^\circ$. By modal reasoning, we then have that $\vdash [U, s][U', s']\varphi \equiv [U, s]([U', s']\varphi)^\circ$. Applying the induction hypothesis to $[U, s]([U', s']\varphi)^\circ$, it then follows

313 that $\vdash [U, s]([U', s']\varphi)^\circ \equiv ([U, s]([U', s']\varphi)^\circ)^\circ$. Since we have that $([U, s]([U', s']\varphi)^\circ)^\circ =$
 314 $([U, s][U', s']\varphi)^\circ$ by the definition of \circ , (iii) follows.

315 We now prove (iv).

$$\begin{aligned}
 & c([U, s][U', s']\varphi)^\circ \\
 = & c([U, s]([U', s']\varphi)^\circ)^\circ && \text{def. of } \circ \\
 < & c([U, s]([U', s']\varphi)^\circ) && \text{IH on } [U, s]([U', s']\varphi)^\circ \\
 = & (4 + c(U)) \cdot c([U', s']\varphi)^\circ && \text{def. of } c \\
 < & (4 + c(U)) \cdot c([U', s']\varphi) && \text{IH on } [U', s']\varphi \\
 = & c([U, s][U', s']\varphi) && \text{def. of } c
 \end{aligned}$$

316 Note that we obtained the two strict inequalities above due to the form of each of
 317 $[U, s]([U', s']\varphi)^\circ$ and $[U, s][U', s']\varphi$. □

318 3 Semantics

319 Having defined the language L_{DETL} and theory T_{DETL} of Dynamic Epistemic Temporal Logic,
 320 we now define the semantics of L_{DETL} .

321 **Definition 3.1.** A *Kripke model* is a tuple $M = (W, R, V)$ for which

- 322 • (W, R) is a Kripke frame for $A \cup \{Y\}$ that will be called the Kripke frame *underlying*
 323 M , and
- 324 • $V : \{p_k : k \in \mathbb{N}\} \rightarrow 2^W$ is a function mapping each propositional letter p_k to a set of
 325 worlds $V(p_k) \subseteq W$.

326 A *world* in M is just a world in the Kripke frame underlying M . Notation: for a Kripke
 327 model M , we write W^M to denote the first element of the tuple M , we write R^M to denote
 328 the second element of the tuple M , and we write V^M to denote the third element of the
 329 tuple M . A *pointed Kripke model* is a pair (M, w) consisting of a Kripke model M and a
 330 world $w \in W^M$ that will be called the *point* of (M, w) .

331 We view a Kripke model as an *epistemic temporal model*, by which we mean that there are
 332 epistemic components capturing what agents consider possible and a temporal component
 333 that accesses features of the past. The worlds W^M of a Kripke model M represent possible
 334 states of affairs. The function R_a^M represents agent a 's uncertainty as to the actual state of
 335 affairs: to have $w' \in R_a^M(w)$ says that whenever w is the actual state of affairs, agent a will
 336 think that w' is one of the possible states of affairs that might be the actual state of affairs.
 337 The function R_Y^M represents a discrete previous-time relation: to have $w' \in R_Y^M(w)$ says that
 338 w' was a possible state of affairs occurring immediately before w . The function V^M is a
 339 *propositional valuation*, which says at which worlds a propositional letter p_k is true. Note
 340 that we do not in general impose any restrictions on the function R^M in a Kripke model
 341 M , though Section 4 shows how we can naturally impose conditions that make the temporal
 342 function R_Y^M and the epistemic functions R_a^M behave in familiar ways.

343 The notion of L_{DETL} -truth extends the standard semantics for Dynamic Epistemic Logic
 344 [2, 3, 4, 10, 17, 19] in the following way.

345 **Definition 3.2** (L_{DETL} -Truth, L_{DETL} -Validity). For a pointed Kripke model (M, w) and
 346 a formula φ , we write $M, w \models_{L_{\text{DETL}}} \varphi$ to mean that φ is *true at* (M, w) , and we write
 347 $M, w \not\models_{L_{\text{DETL}}} \varphi$ to mean that φ is not true at (or *false at*) (M, w) . The notion of truth
 348 of a formula at a pointed Kripke model is defined by the following induction on formula
 349 construction.

- 350 • $M, w \not\models_{L_{\text{DETL}}} \perp$ and $M, w \models_{L_{\text{DETL}}} \top$.
- 351 • $M, w \models_{L_{\text{DETL}}} p_k$ means that $w \in V^M(p_k)$.
- 352 • $M, w \models_{L_{\text{DETL}}} \varphi \star \psi$ means that $M, w \models_{L_{\text{DETL}}} \varphi$ star $M, w \models_{L_{\text{DETL}}} \psi$, where “star” is to
 353 be replaced by the (mathematical) English reading for the binary Boolean connective
 354 \star .²
- 355 • $M, w \models_{L_{\text{DETL}}} \neg\varphi$ means that $M, w \not\models_{L_{\text{DETL}}} \varphi$.
- 356 • $M, w \models_{L_{\text{DETL}}} [a]\varphi$ means that $M, x \models_{L_{\text{DETL}}} \varphi$ for each $x \in R_a^M(w)$.
- 357 • $M, w \models_{L_{\text{DETL}}} [U, s]\varphi$ means that if $M, w \models_{L_{\text{DETL}}} \mathbf{p}^U(s)$, then $M[U], (w, s) \models_{L_{\text{DETL}}} \varphi$,
 358 where the model $M[U]$ is defined as follows.

$$W^{M[U]} \quad := \quad \{(x, t) \in W^M \times W^U : M, x \models_{L_{\text{DETL}}} \mathbf{p}^U(t)\}$$

for $a \in A$,

$$R_a^{M[U]}(x, t) \quad := \quad \{(y, u) \in W^{M[U]} : y \in R_a^M(x) \text{ and } u \in R_a^U(t)\}$$

$$R_Y^{M[U]}(x, t) \quad := \quad \{(y, u) \in W^{M[U]} : y \in R_Y^M(x) \text{ and } u \in R_Y^U(t)\} \cup \\ \{(y, u) \in W^{M[U]} : y = x \text{ and } u \in R_Y^U(t)\}$$

$$V^{M[U]}(p_k) \quad := \quad \{(x, t) \in W^{M[U]} : M, x \models_{L_{\text{DETL}}} p_k\}$$

359 To say that a formula φ is *valid in* a Kripke model M , written $M \models_{L_{\text{DETL}}} \varphi$, means that
 360 $M, w \models_{L_{\text{DETL}}} \varphi$ for each world $w \in W^M$. To say that a formula φ is *valid*, written $\models_{L_{\text{DETL}}} \varphi$,
 361 means that $M \models_{L_{\text{DETL}}} \varphi$ for each Kripke model M . When it ought not cause confusion, we
 362 may omit the subscript “ L_{DETL} ” when writing $\models_{L_{\text{DETL}}}$.

363 Given a pointed Kripke model (M, w) representing a multi-agent situation and a pointed
 364 update frame (U, s) with $M, w \models \mathbf{p}^U(s)$, the pointed Kripke model $(M[U], (w, s))$ represents
 365 the situation after the occurrence of the update described by $[U, s]$. According to Definition
 366 3.2, a world (x, t) must satisfy the property that $M, x \models \mathbf{p}^U(t)$. The set $\{x \in W^M : M, x \models$
 367 $\mathbf{p}^U(t)\}$ of worlds x in M that satisfy $\mathbf{p}^U(t)$ intuitively represents the set of worlds in M at
 368 which the formula $\mathbf{p}^U(t)$ can truthfully be communicated—these are the worlds at which t
 369 can take place.

370 For each $a \in A$, Definition 3.2 tells us that the relation $R_a^{M[U]}$ is determined by two factors:
 371 agent a ’s uncertainty as to which world was the case before the communication (represented

²Read \rightarrow as “implies,” read \vee as “or,” read \wedge as “and,” and read \equiv as “if and only if.”

372 by R_a^M) and agent a 's uncertainty as to which communication has occurred (represented by
 373 R_a^U). In particular, suppose $(x', t') \in R_a^{M[U]}(x, t)$. Then if the communication corresponding
 374 to t actually occurred at world x , then agent a will think it possible that the communication
 375 corresponding to t' occurred at world x' .

376 According to Definition 3.2, the relation $R_Y^{M[U]}$ is determined by two factors. The first
 377 is the interaction between the relations R_Y^U and R_Y^M , which adds pairs to $R_Y^{M[U]}$ just as the
 378 interaction between R_a^U and R_a^M did to $R_a^{M[U]}$ for $a \in A$. The second factor is the relation
 379 $R_{\underline{Y}}$: if there is a \underline{Y} -arrow from state t to state t' in U , then there will be a Y -arrow from
 380 world (x, t) to world (x, t') in $M[U]$. The presence of a \underline{Y} -arrow from t to t' in U thus says
 381 that the communication corresponding to t' is to be thought of as occurring one time-step
 382 before the communication corresponding to t . This addition to the standard definition of
 383 updates in Dynamic Epistemic Logic [2, 3, 4, 10, 17, 19] allows us to control how an update
 384 affects the time of worlds in the model $M[U]$.

385 Finally, we see that the valuation $V^{M[U]}$ after the update simply inherits its truth con-
 386 ditions from the valuation V^M before the update, making our updates *purely temporal-*
 387 *epistemic*.

388 **Example 3.3.** Figure 4 illustrates the way in which the function $M \mapsto M[U]$ defined in
 389 Definition 3.2 produces a model $M[U]$ from an existing Kripke model M and an update frame
 390 U . Suppose the following figure represents our pointed update frame (U, t) , representing the
 391 asynchronous private announcement of p to only some of the agents G in the group A . The
 392 other agents in $A - G$ believe that nothing happened. The Y -arrow from t to s represents
 393 the fact that s is one time step behind t . State s can be thought of as a kind of non-event,
 394 in which time does not pass; it can also occur at every world in W^M . Thus worlds (x, s) in
 395 an updated model will be seen as copies of worlds in M . Worlds (y, t) in the updated model
 396 will represent future states.

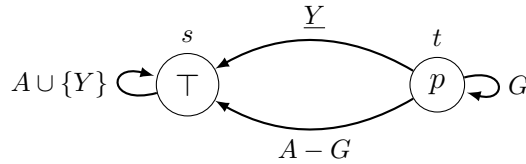


Figure 3: Update frame U for the asynchronous private announcement of p to group $G \subseteq A$

397 Now, consider the result of this private announcement taking place in a pointed Kripke
 398 model (M, w) (on the left in Figure 4) in which our agents do not know whether p holds.
 399 And the update frame $M[U]$ (on the right in Figure 4) represents the updated model $M[U]$
 400 Notice that we have a copy of M in $M[U]$, with all relations preserved. (w, s) and (v, s) in
 401 $W^{M[U]}$ can be thought of as counterparts of w and v in W^M , respectively. However, the only
 402 world in W^M at which t can be announced is w . Thus, (w, t) is the only new world created,
 403 and since there is a \underline{Y} -arrow from t to s in U , there is a corresponding Y -arrow from (w, t)
 404 to (w, s) in $M[U]$.

405 The Y -arrows leaving a world represent a way to count time, or equivalently in our
 406 discussion, the number of events which have taken place. In our example, (w, t) , at which

407 one event has taken place (or the time is 1), becomes the actual world in $M[U]$. But at that
 408 world, agents in $A - G$ believe themselves to be at (w, s) in which no events have taken place
 409 (the time is 0). We can say informally that agents in $A - G$ have their local clocks out of
 410 sync with the global clock, since they believe that the time is 0, when in fact the time is 1.
 411 This notion of time will be spelled out more formally, however, in Section 4.

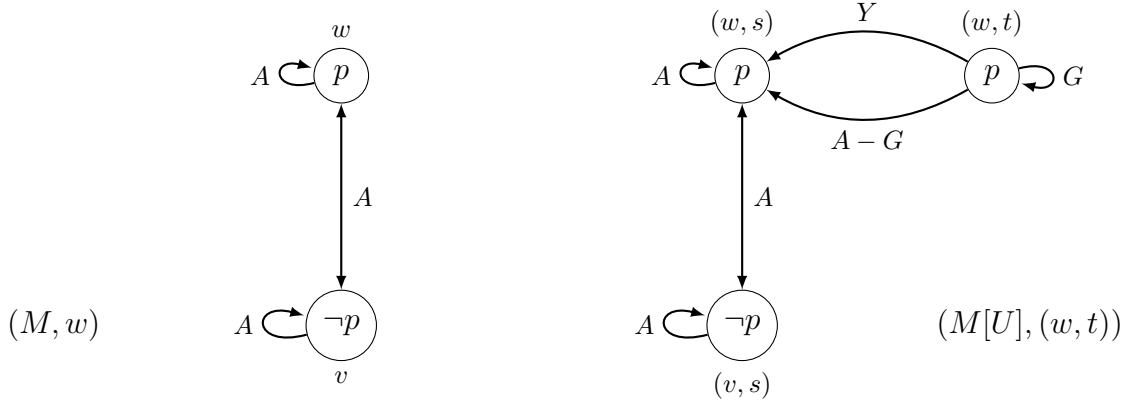


Figure 4: (M, w) is a pointed Kripke model in which the agents do not know whether p . $(M[U], (w, t))$ is an updated pointed Kripke model in which agents in G know that p holds, but agents in $A - G$ do not even know that an event took place.

412 We now define the notion of *executability*, which is employed in Section 4.

413 **Definition 3.4** (Executability). Let M be a Kripke model and U be an update frame. To
 414 say that U is *executable in M* means that there is a world $w \in W^M$ and a state $s \in W^U$
 415 such that $M, w \models \mathbf{p}^U(s)$.

416 **Theorem 3.5** (Correctness). For each formula φ , we have $\vdash \varphi$ if and only if $\models \varphi$.

417 *Proof.* Let us first show the *Soundness* of T_{DETL} : $\vdash \varphi$ implies $\models \varphi$. The argument is by
 418 induction on the length of derivations in T_{DETL} . In the base case, we are to show that every
 419 T_{DETL} -axiom is valid. We check each axiom in turn.

- 420 • Axiom CL is valid; that is, each axiom of Classical Propositional Logic is valid.

421 This follows by standard truth-table arguments.

- 422 • Axioms K_a and K_Y are valid.

423 This follows by the standard modal argument [5].

- 424 • Axiom UA is valid: $\models [U, s]q \equiv (\mathbf{p}^U(s) \rightarrow q)$ for $q \in \{p_k, \perp, \top\}$.

425 Given a pointed Kripke model (M, w) , we argue that $M, w \models [U, s]q$ if and only if
 426 $M, w \models \mathbf{p}^U(s) \rightarrow q$. First, note that in case $M, w \not\models \mathbf{p}^U(s)$, then the result follows
 427 immediately by the definition of truth. So let us assume that $M, w \models \mathbf{p}^U(s)$, in which
 428 case it suffices to show that $M[U], (w, s) \models q$ if and only if $M, w \models q$. If $q \in \{\perp, \top\}$,

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then the result is immediate by the definition of truth. So let us assume that $q = p_k$. We then have that $M, w \models p_k$ if and only if $w \in V^M(p_k)$, but the latter is equivalent to $(w, s) \in V^{M[U]}(p_k)$ by the definition of truth. The result follows.

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- Axiom $U\star$ is valid: $\models [U, s](\varphi \star \psi) \equiv ([U, s]\varphi \star [U, s]\psi)$.

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Given a pointed Kripke model (M, w) , we argue that $M, w \models [U, s](\varphi \star \psi)$ if and only if $M, w \models [U, s]\varphi \star [U, s]\psi$. First, note that in case $M, w \not\models \mathbf{p}^U(s)$, then we have by the definition of truth that $M, w \models [U, s](\varphi \star \psi)$, $M, w \models [U, s]\varphi$, and $M, w \models [U, s]\psi$; if this is so, then the result follows by an inspection of each of the cases $\star \in \{\rightarrow, \vee, \wedge, \equiv\}$. So we may assume that $M, w \models \mathbf{p}^U(s)$. It therefore suffices for us to show that $M[U], (w, s) \models \varphi \star \psi$ if and only if $M[U], (w, s) \models \varphi$ star $M[U], (w, s) \models \psi$, where “star” is to be replaced by the English reading of the binary Boolean connective \star . But this follows immediately by the definition of truth. The result follows.

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- Axiom $U\neg$ is valid: $\models [U, s]\neg\varphi \equiv (\mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi)$.

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Given a pointed Kripke model (M, w) , we argue that $M, w \models [U, s]\neg\varphi$ if and only if $M, w \models \mathbf{p}^U(s) \rightarrow \neg[U, s]\varphi$. First, note that the result follows by the definition of truth in case $M, w \not\models \mathbf{p}^U(s)$. So let us assume that $M, w \models \mathbf{p}^U(s)$. It then suffices for us to show that $M[U], (w, s) \not\models \varphi$ if and only if $M, w \not\models [U, s]\varphi$. But $M, w \not\models [U, s]\varphi$ is equivalent under our assumption $M, w \models \mathbf{p}^U(s)$ to the assertion $M[U], (w, s) \not\models \varphi$. The result follows.

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- Axiom $U[a]$ is valid: $\models [U, s][a]\varphi \equiv (\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_a^U(s)} [a][U, s']\varphi)$ for $a \in A$.

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Given a pointed Kripke model (M, w) , we argue that $M, w \models [U, s][a]\varphi$ if and only if $M, w \models \mathbf{p}^U \rightarrow \bigwedge_{s' \in R_a^U(s)} [a][U, s']\varphi$. As in the previous cases, the result follows in case $M, w \not\models \mathbf{p}^U(s)$, so we will assume that $M, w \models \mathbf{p}^U(s)$. It then suffices for us to show that $M[U], (w, s) \models [a]\varphi$ if and only if $M, w \models [a][U, s']\varphi$ for each $s' \in R_a^U(s)$. By the definition of truth, to have $M[U], (w, s) \models [a]\varphi$ means that for each $(w', s') \in W^{M[U]}$ with $w' \in R_a^M(w)$ and $s' \in R_a^U(s)$, we have that $M[U], (w', s') \models \varphi$. This is equivalent to the statement that for each $w' \in R_a^M(w)$ and each $s' \in R_a^U(s)$, if we have that $M, w' \models \mathbf{p}^U(s')$, then we have that $M[U], (w', s') \models \varphi$. But this is itself equivalent to the statement that $M, w \models [a][U, s']\varphi$ for each $s' \in R_a^U(s)$. The result follows.

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- Axiom $U[Y]$ is valid: $\models [U, s][Y]\varphi \equiv (\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y][U, s']\varphi) \wedge (\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s)} [U, s']\varphi)$.

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Given a pointed Kripke model (M, w) , we argue that $M, w \models [U, s][Y]\varphi$ if and only if both $M, w \models \mathbf{p}^U \rightarrow \bigwedge_{s' \in R_Y^U(s)} [Y][U, s']\varphi$ and $M, w \models \mathbf{p}^U \rightarrow \bigwedge_{s' \in R_Y^U(s)} [U, s']\varphi$. As in previous cases, $M, w \not\models \mathbf{p}^U(s)$ implies the result, so we assume that $M, w \models \mathbf{p}^U(s)$. It then suffices for us to prove that $M[U], (w, s) \models [Y]\varphi$ if and only if for each $s' \in R_Y^U(s)$ and each $t' \in R_Y^U(s)$, we have $M, w \models [Y][U, s']\varphi$ and $M, w \models [U, t']\varphi$. By the definition of truth, to have $M[U], (w, s) \models [Y]\varphi$ means that for each $(w', s') \in W^{M[U]}$ with $w' \in R_Y^M(w)$ and $s' \in R_Y^U(s)$, we have $M[U], (w', s') \models \varphi$. But this is equivalent by the definition of truth to the statement that for each $w' \in R_Y^M(w)$ and each $s' \in W^U$ with $M, w \models \mathbf{p}^U(s')$, we have that $M[U], (w', s') \models \varphi$ in case $s' \in R_Y^U(s)$ and that

468 $M[U], (w, s') \models \varphi$ in case $s' \in R_Y^U(s)$. But the latter statement is equivalent by the
 469 definition of truth to the statement that for each $s' \in R_Y^U(s)$ and each $t' \in R_Y^U(s')$, we
 470 have $M, w \models [Y][U, s']\varphi$. The result follows.

471 This completes the base cases of Soundness. For the induction cases, we are to show that
 472 validity is preserved under each of the rules of inference of T_{DETL} . We consider each rule in
 473 turn.

474 • Rule MP preserves validity: $\models \varphi \rightarrow \psi$ and $\models \varphi$ together imply that $\models \psi$.

475 This follows by the standard modal argument [5].

476 • Rule MN preserves validity: $a \in A \cup \{Y\}$ and $\models \varphi$ together imply that $\models [a]\varphi$.

477 This follows by the standard modal argument [5].

478 • Rule UN preserves validity: $\models \varphi$ implies $\models [U, s]\varphi$.

479 Assume $\models \varphi$ and let (M, w) be a pointed Kripke model. If $M, w \not\models \mathbf{p}^U(s)$, then we have
 480 $M, w \models [U, s]\varphi$. If $M, w \models \mathbf{p}^U(s)$, then we have $M[U], (w, s) \models \varphi$ by our assumption
 481 $\models \varphi$, and so $M, w \models [U, s]\varphi$. It follows that $M, w \models [U, s]\varphi$ no matter whether
 482 $M, w \models \mathbf{p}^U(s)$. Since (M, w) was chosen arbitrarily, we have shown that $\models [U, s]\varphi$.

483 This completes the induction. Conclusion: $\vdash \varphi$ implies $\models \varphi$.

484 We now prove *Completeness* of T_{DETL} : $\models \varphi$ implies $\vdash \varphi$. Completeness is proved by a
 485 canonical model argument. We begin with some preliminary definitions. The *conjunction*
 486 of a finite set S of formulas, written $\bigwedge S$, is defined by setting $\bigwedge S := \top$ if $S = \emptyset$ and
 487 $\bigwedge S := \bigwedge_{\psi \in S} \psi$ if $S \neq \emptyset$. To say that a set S of formulas is *consistent* means that for no
 488 finite subset $S' \subseteq S$ do we have that $\vdash \neg \bigwedge S'$. The negation of “consistent” is “inconsistent.”
 489 To say that a set S of formulas is *maximal consistent* means that S is consistent and that
 490 adding to S a formula not already in S produces an inconsistent set. By a Lindenbaum
 491 Argument, every consistent set of formulas may be extended to a maximal consistent set of
 492 formulas. That there exists a consistent set of formulas follows by Soundness.

493 We define a tuple $M = (W, R, V)$ called the *canonical model* as follows. W is the set of
 494 all maximal consistent sets of formulas. For each $w \in W$ and each $a \in A \cup \{Y\}$, we define
 495 $w^a := \{\varphi \in L_{\text{DETL}} \mid [a]\varphi \in w\}$ and then define $R : (A \cup \{Y\}) \rightarrow (W \rightarrow 2^W)$ by setting
 496 $R_a(w) := w^a$. Finally, we define $V : \{p_k \mid k \in \mathbb{N}\} \rightarrow 2^W$ by setting $V(p_k) := \{w \in W \mid p_k \in$
 497 $w\}$. We observe that the canonical model is in fact a Kripke model; after all, $W \neq \emptyset$ follows
 498 by the fact that there exists a consistent set of formulas.

499 The canonical model M satisfies the *Truth Lemma for L_{ETL}* : for each $w \in W^M$ and each
 500 L_{ETL} -formula φ , we have $\varphi \in w$ if and only if $M, w \models \varphi$. The proof of the Truth Lemma for
 501 L_{ETL} follows the standard argument in modal logic [5].

502 We now argue for Completeness by contraposition: $\not\models \varphi$ implies $\not\vdash \varphi$. So assume that $\not\models \varphi$.
 503 It follows by the Reduction Theorem (Theorem 2.11) and propositional reasoning that $\not\models \varphi^\circ$.
 504 We then have that $\{\neg\varphi^\circ\}$ is consistent and so may be extended to a maximal consistent set
 505 $w \in W^M$, where M is the canonical model. Since $\neg\varphi^\circ \in w$ and $\neg\varphi^\circ \in L_{\text{ETL}}$, it follows by
 506 the Truth Lemma for L_{ETL} that $M, w \models \neg\varphi^\circ$ and thus that $M, w \not\models \varphi^\circ$ by the definition of
 507 truth. Applying Soundness and the Reduction Theorem (Theorem 2.11), we therefore have

508 that $M, w \not\models \varphi$. Since (M, w) is a pointed Kripke model, we have shown that $\not\models \varphi$ implies
509 $\not\models \varphi$. This completes the proof of Completeness. Conclusion: T_{DETL} is sound and complete
510 with respect to its intended semantics. \square

511 4 Properties and Preservation

512 In this section, we define several properties of Kripke models and update frames and then
513 study sufficient conditions for the preservation of these properties after the occurrence of an
514 update.

515 **Definition 4.1** (*T-Runs, T-Histories, T-Depth*). Fix a symbol $T \in \{Y, \underline{Y}\}$ and let $F =$
516 (W, R) be a Kripke frame for $A \cup \{Y, T\}$. A *T-run* (in F) is a finite nonempty sequence
517 $\{w_i\}_{i=0}^n$ of worlds in F satisfying the property that $n \in \mathbb{N}$ and for each $i \in \mathbb{N}$ with $i < n$, we
518 have that $w_{i+1} \in R_T^F(w_i)$. We say that a *T-run* $\{w_i\}_{i=0}^n$ *begins at* w_0 and *ends at* w_n . The
519 *length* of a *T-run* $\{w_i\}_{i=0}^n$ is defined as the number n . (Observe that the length of a *T-run*
520 is one less than the number of worlds that make up the *T-run*.)

521 A *prefix* of a *T-run* $\{w_i\}_{i=0}^n$ is a *T-run* $\{w_i\}_{i=0}^m$ with $m \leq n$. (Note that each *T-run* is a
522 prefix of itself.) A *suffix* of a *T-run* $\{w_i\}_{i=0}^n$ is a *T-run* $\{v_i\}_{i=0}^m$ with $m \leq n$ and $v_i = w_{i+(n-m)}$
523 for each $i \in \mathbb{N}$ with $i \leq m$. (Note that each *T-run* is a suffix of itself.) To say that a prefix
524 or a suffix σ' of σ is *proper* means that $\sigma' \neq \sigma$.

525 To say that a *T-run* σ' *end-extends* a *T-run* σ means that σ is a (not necessarily proper)
526 prefix of σ' . (Note that each *T-run* end-extends itself.) To say that a *T-run* σ is *end-maximal*
527 (in F) means that no *T-run* in F end-extends σ .

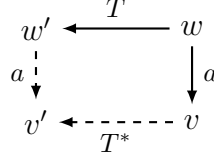
528 A *T-history* (in F) is a *T-run* in F that is end-maximal. (Note that a suffix of a *T-*
529 *history* is itself a *T-history*.) A world appearing at the end of a *T-history* in F is said to
530 be *T-terminal* (in F). We define a function $d_T^F : W^F \rightarrow \mathbb{N} \cup \{\infty\}$ as follows: if there is a
531 maximum $n \in \mathbb{N}$ such that there is a *T-history* in F of length n that begins at w , then $d_T^F(w)$
532 is n ; otherwise, if no such maximum $n \in \mathbb{N}$ exists, then $d_T^F(w)$ is ∞ . We will call $d_T^F(w)$ the
533 *T-depth of* w .

534 **Definition 4.2.** Fix a symbol $T \in \{Y, \underline{Y}\}$ and let $F = (W, R)$ be a Kripke frame for
535 $A \cup \{Y, T\}$.

- 536 • *T-Depth-Defined (T-DD)*. To say that F is *T-depth-defined (T-DD)* means that for
537 each world w in F , we have that $d_T^F(w) \neq \infty$.³
- 538 • *Non-T-Branching*. To say that F is *non-T-branching* means that for each $w \in W^F$,
539 the set $R_T^F(w)$ has at most one member.
- 540 • *T-Synchronous*. If F is *T-DD*, then to say that F is *T-synchronous* means that for
541 each $a \in A$, each $w \in W^F$, and each $w' \in R_a^F(w)$, we have that $d_T^F(w') = d_T^F(w)$. The
542 negation of “*T-synchronous*” is *T-asynchronous*.

³We observe that if F is *T-depth-defined*, then F is *T-converse well-founded* (that is, for every nonempty set S of worlds in F , there is a nonempty subset $S' \subseteq S$ such that for each $w \in S'$, the unique *T-run* in F that begins at w has length zero). However, if F is *T-converse well-founded*, it need not be the case that F is also *T-depth-defined*. So the notion of *T-depth-definedness* is strictly stronger than the notion of *T-converse well-foundedness*.

- 543 • *T-Memory-Preserving*. To say that F is *T-memory-preserving* means that for each
 544 $a \in A$, each $w \in W^F$, each $v \in R_a^F(w)$, and each $w' \in R_T^F(w)$, there is a world
 545 $v' \in R_a^M(w') \cap (R_T^F)^*(v)$, where $(R_T^F)^*$ is the reflexive transitive closure of R_T^F .⁴ Picture:



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547 Given a T -synchronous frame, the well-known property of *Perfect Recall (for T)* [15,
 548 16] is just T -memory-preservation.

549 Convention: for tuples J having a Kripke frame (W^J, R^J) underlying J , any use of a
 550 property or concept from Definition 4.1 or Definition 4.2 in reference to J is meant to be
 551 a use of that property or concept in reference to the Kripke frame (W^J, R^J) underlying J .
 552 Example: for an update frame U , the expression “ \underline{Y} -run in U ” is to be identified with the
 553 expression “ \underline{Y} -run in (W^U, R^U) .”

554 **Definition 4.3** (Kripke Model Properties). Let M be a Kripke model.

- 555 • *Synchronicity (under Y-DD)*. If M is Y -DD, then to say that M is *synchronous* means
 556 that M is Y -synchronous. The negation of “synchronous” is *asynchronous*.
- 557 • *Non-Past-Branching*. To say that M satisfies *non-past-branching* means that M is
 558 non- Y -branching.
- 559 • *Forest-like*. To say that M is *forest-like* means that M is Y -DD and non-past-
 560 branching.
- 561 • *Memory-Preserving (under non-past-branching)*. To say that M is *memory-preserving*
 562 means that M is Y -memory-preserving.⁵ As above, in a synchronous Kripke model,
 563 *Perfect Recall* [15, 16] is memory preservation.

564 **Definition 4.4** (Update Frame Properties and Concepts). Let U be an update frame.

- 565 • *Path-Preserving*. A *path-preserving run* (in U) is a \underline{Y} -run $\{s_i\}_{i=0}^n$ in U satisfying the
 566 property that for each $i \in \mathbb{N}$ with $i < n$, we have $\models \mathbf{p}^U(s_i) \rightarrow \mathbf{p}^U(s_{i+1})$. To say that U
 567 is a *path-preserving update frame* means that each \underline{Y} -run in U is path-preserving.
- 568 • *Depth-Respecting (under Y-DD)*. If U is \underline{Y} -DD, then to say that U is *depth-respecting*
 569 means that for each $s \in W^U$ and each $s' \in R_Y^U(s)$, we have that $d_Y^U(s') \leq d_Y^U(s)$.

⁴The reflexive transitive closure $R^* : A \rightarrow (W \rightarrow 2^W)$ of $R : A \rightarrow (W \rightarrow 2^W)$ is given as follows:
 $v' \in R^*(v)$ means that there is a finite possibly empty sequence $\{v_i\}_{i=1}^n$ with $n \in \mathbb{N}$ such that $v_1 = v$,
 $v_n = v'$, and $v_{i+1} \in R(v_i)$ for each $i \in \mathbb{N}$ with $1 \leq i < n$.

⁵Informally, in a memory-preserving Kripke model, an agent considers a world possible at time t only if
 that world was a possible future at time $t - 1$.

- 570 • *Past State, Past-Preserving.* A *past state* is a state $s \in W^U$ satisfying the property
571 that $\mathbf{p}^U(s) = \top$, that $R_{\underline{Y}}^U(s) = \emptyset$, and that $R_a^U(s) = \{s\}$ for each $a \in A \cup \{Y\}$. To say
572 that U is *past-preserving* means U is \underline{Y} -DD and path-preserving and that every \underline{Y} -run
573 in U can be end-extended to a \underline{Y} -history in U that ends at a past state.
- 574 • *Non-Past-Splitting.* To say that U is *non-past-splitting* means that $R_{\underline{Y}}^U(s) \cup R_Y^U(s)$ has
575 at most one element and $R_{\underline{Y}}^U(s) \cap R_Y^U(s) = \emptyset$ for each $s \in W^U$.
- 576 • *Memory-Respecting (under \underline{Y} -DD and non-past-splitting).* To say that U is *memory-*
577 *respecting* means that U is \underline{Y} -memory-preserving.

578 Having defined these properties, we investigate their preservation under the presence of
579 updates in the following two theorems. Theorem 4.5 concerns the behavior of past states in
580 update frames, and Theorem 4.6 concerns the preservation of properties in Kripke models.

581 **Theorem 4.5** (Past State Theorem). Each of the following holds.

- 582 1. If s is a past state in U , then for each $\varphi \in L_{\text{DETL}}$ and each $w \in W^M$, we have that
583 $M[U], (w, s) \models \varphi$ if and only if $M, w \models \varphi$.
- 584 2. Suppose U is past-preserving, $s \in W^U$, and $w \in W^M$ satisfies $M, w \models \mathbf{p}^U(s)$. Then
585 there is an $n \in \mathbb{N}$ such that for each $\varphi \in L_{\text{DETL}}$, we have that $M[U], (w, s) \models \langle Y \rangle^n \varphi$ if
586 and only if $M, w \models \varphi$.

587 *Proof.* We first prove Item 1. Let s be a past state in U , and let $w \in W^M$. We will prove
588 that $M[U], (w, s) \models \varphi$ if and only if $M, w \models \varphi$ by induction on L_{ETL} -formulas. By Theorem
589 2.11, we have provable equivalence between L_{DETL} and L_{ETL} , so we can restrict our attention
590 to the simpler case. For the base case, let φ be a proposition letter p . This case follows
591 immediately from the definition of $V^{M[U]}(p)$. So let us proceed with the inductive case.

592 Suppose the result holds for φ . The Boolean cases are straightforward, so we will consider
593 only the $[a]$ modality. Suppose $M, w \models [a]\varphi$, and let $(v, t) \in R_a^{M[U]}(w, s)$. This implies
594 $v \in R_a^M(w)$ and $t \in R_a^U(s)$. This latter fact implies $t = s$, since s is a past state. So we know
595 $M, v \models \varphi$, and by the IH, $M[U](v, t) \models \varphi$, which implies $M[U](w, s) \models [a]\varphi$.

596 Conversely, assume $M[U], (w, s) \models [a]\varphi$. Since s is a past state, $\mathbf{p}^U(s) = \top$ and $R_a^U(s) =$
597 $\{s\}$, so $(v, s) \in W^{M[U]}$ iff $v \in R_a^M(w)$. Now for such a v , we know $M[U], (v, s) \models \varphi$, so by
598 the IH, $M, v \models \varphi$. Thus $M, w \models [a]\varphi$. This completes the proof of Item 1.

599 We now prove Item 2. Let U be past-preserving, $s \in W^U$, and $M, w \models \mathbf{p}^U(s)$. Since
600 U is past-preserving, every \underline{Y} -run in U can be end-extended to a path-preserving \underline{Y} history
601 $\{s_i\}_{i=0}^n$ such that s_n is a past state. We will show that the length of the run is the n required
602 by the theorem. If $n = 0$, then s is already a past state, and the theorem holds by 1. Now
603 suppose that $m < n$ such that for any φ , $M[U](w, s_m) \models \langle Y \rangle^m \varphi$ iff $M, w \models \varphi$. Now we will
604 consider s_{m+1} . Notice that since $\{s_i\}_{i=0}^n$ is path-preserving, $\models \mathbf{p}^U(s_m) \rightarrow \mathbf{p}^U(s_{m+1})$. Thus,
605 $((w, s_m), s_{m+1}) \in R^{M[U][U']}$. Then $M[U][U']((w, s_m), s_{m+1}) \models \langle Y \rangle^{m+1} \varphi$ iff $M[U](w, s_m) \models$
606 $\langle Y \rangle^m \varphi$ iff $M, w \models \varphi$. \square

607 Theorem 4.5 tells us that past states play the role of “maintaining a link to the past”
608 within past-preserving update frames. In particular, if s is a past state, then the submodel

609 of $M[U]$ consisting of the worlds of the form (w, s) for some world $w \in W^M$ is L_{DETL} -
610 indistinguishable from the Kripke model M itself. So the operation $(M, w) \mapsto (M[U], (w, s))$
611 retains a copy of the “past” state of affairs (M, w) . Furthermore, if U is past-preserving,
612 then from any world in $W^{M[U]}$, there is a finite sequence of Y -arrows that leads back to this
613 “past” state of affairs, thereby “maintaining a link to the past.”

614 Let us now examine the preservation of properties of the Kripke model M in the presence
615 of the operation $M \mapsto M[U]$.

616 **Theorem 4.6** (Preservation Theorem). Let U be an update frame that is executable in a
617 Kripke model M .

- 618 1. (Y -DD Preservation). Y -DD. If M is Y -DD and U is \underline{Y} -DD and depth-respecting,
619 then $M[U]$ is Y -DD.
- 620 2. (Synchronicity Preservation). If M is synchronous (and Y -DD) and U is depth-
621 respecting, past-preserving, and \underline{Y} -synchronous, then $M[U]$ is synchronous.
- 622 3. (Non-Past-Branching Preservation). If M is non-past-branching and U is non-past-
623 splitting, then $M[U]$ is non-past-branching.
- 624 4. (Forest-like Preservation). If M is forest-like and U is \underline{Y} -DD and non-past-splitting,
625 then $M[U]$ is forest-like.
- 626 5. (Memory Preservation). If M is memory-preserving (and Y -DD and non-past-branching),
627 and if U is past-preserving, non-past-splitting, and memory-respecting, then $M[U]$ is
628 memory-preserving.
- 629 6. (S5 Preservation). If M is such that R_a^M is an equivalence relation for every agent
630 $a \in A$, and U is such that R_a^U is an equivalence relation for every $a \in A$, then $M[U]$ is
631 such that $R_a^{M[U]}$ is an equivalence for every $a \in A$.

632 *Proof. 1. Y -DD Preservation:* Suppose that M is a Y -DD Kripke model, and U is a
633 \underline{Y} -DD update frame. Then given a world $(w, s) \in W^{M[U]}$, we prove our desired result by
634 double induction, with an outer induction on the depth of w and an inner induction on the
635 depth of s .

636 **2. Synchronicity Preservation:** The proof of this follows directly from the fol-
637 lowing claim: suppose U is \underline{Y} -depth-defined, depth-respecting, path-preserving, and past-
638 preserving. Then $d_Y^{M[U]}(w, s) = d_Y^M(w) + d_{\underline{Y}}^U(s)$.

639 We prove this by induction on $n = d_Y^M(w) + d_{\underline{Y}}^U(s)$. For the base case ($n = 0$), we have
640 $d_Y^M(w) = 0$ and $d_{\underline{Y}}^U(s) = 0$, and hence w is a Y -history and s is a \underline{Y} -history. As s is a
641 \underline{Y} -history, s must be a past state, due to the condition that U is past preserving. The only
642 way then for a world to be in $R_Y^{M[U]}(w, s)$ is for there to be a world in $R_Y^M(w)$. But as
643 $d_Y^M(w) = 0$, there are no worlds in $R_Y^M(w)$. Thus $d_Y^{M[U]}(w, s) = 0$.

644 For the inductive hypothesis, assume that $d_Y^{M[U]}(w, s) = d_Y^M(w) + d_{\underline{Y}}^U(s)$ whenever $d_Y^M(w) +$
645 $d_{\underline{Y}}^U(s) \leq n$. Now suppose that $(w, s) \in W^{M[U]}$ and $d_Y^M(w) + d_{\underline{Y}}^U(s) = n + 1$.

646 We first show that $d_Y^{M[U]}(w, s) \geq n + 1$. Let us first suppose that $d_{\underline{Y}}^U(s) = k + 1$ for $k \geq 0$.
647 Then there is $s' \in R_{\underline{Y}}^U(s)$ for which $d_{\underline{Y}}^U(s') = k$. Since U is path preserving, $(w, s') \in W^{M[U]}$,

648 and by the inductive hypothesis, $d_Y^{M[U]}(w, s') = n$. As $(w, s)R_Y^{M[U]}(w, s')$, $d_Y^{M[U]}(w, s) \geq n+1$.
649 Suppose instead that $d_Y^U(s) = 0$ (and hence $d_Y^M(w) = n+1$). Then there is $w' \in R_Y^M(w)$
650 for which $d^M(w') = n$, and by the inductive hypothesis, $d_Y^{M[U]}(w', s) = n$. Since U is past
651 preserving, s is past state, and hence $(w, s)R_Y^{M[U]}(w', s)$. Thus $d_Y^{M[U]}(w, s) \geq n+1$.

652 Next we show that $d_Y^{M[U]}(w, s) \leq n+1$. Suppose that $(w, s)R_Y^{M[U]}(w', s')$. If $w = w'$,
653 then since $w' \notin R_Y^M(w)$, we must have that $sR_Y^U s'$. As $d_Y^U(s') < d_Y^U(s)$, it is the case that
654 $d_Y^M(w') + d_Y^U(s') \leq n$, and we can apply the inductive hypothesis to get that $d_Y^{M[U]}(w', s') \leq n$.
655 If $w \neq w'$, then $d_Y^M(w') < d_Y^M(w)$. Since U is depth respecting, $d_Y^U(s') \leq d_Y^U(s)$, and by the
656 inductive hypothesis, $d_Y^{M[U]}(w', s') \leq n$. Hence $d_Y^{M[U]}(w, s) \leq n+1$. This concludes the proof
657 of the claim.

658 If $(w, s)R_a^{M[U]}(w', s')$, then $wR_a^M w'$ and $sR_a^U s'$. By synchronicity of M , $d_Y^M(w) = d_Y^M(w')$
659 and by synchronicity of U , $d_Y^U(s) = d_Y^U(s')$. By the claim, $d_Y^{M[U]}(w, s) = d_Y^{M[U]}(w', s')$.

660 **3. Non-Past-Branching Preservation:** Suppose that U is non-past-splitting and
661 that M is non-past-branching. Suppose also that $(w, s)R_Y^{U[M]}(w', s')$ and $(w, s)R_Y^{U[M]}(w'', s'')$.
662 Note that sRs' and sTs'' for $S, T \in \{R_Y^U, R_Y^U\}$. Thus $s' = s''$, for otherwise $R_Y^U \cup R_Y^U$ is not
663 a partial function, thus violating a property of U being non-past-splitting. Then by our
664 assumption of the relations in U , $R = T$. If $R = R_Y^U$, then $w = w' = w''$. If $R = R_a^U$, then by
665 partial functionality of R_a^M , $w' = w''$. Thus $(w', s') = (w'', s'')$.

666 **4. Forest-like Preservation:** Suppose U is \underline{Y} -DD and non-past-splitting. Suppose
667 also that M is forest-like. Then M is Y -DD and non-past-branching. As U is \underline{Y} -DD, by part
668 (1) of this theorem, $M[U]$ is Y -DD. As U is non-past-splitting, by part (3) of this theorem,
669 $M[U]$ is non-past-branching, and hence $M[U]$ is forest-like.

670 **5. Memory Preservation:** Suppose U is past-preserving, non-past-splitting, and
671 memory-respecting, and suppose that M is Y -DD, non-past-branching, and memory-preserving.
672 Suppose that $(w, s)R_Y^{M[U]}(w', s')$ and $(w, s)R_a^{M[U]}(v, t)$. We consider two cases: s is a past
673 state, and s is not a past state. In the first case, $wR_Y^M w'$ and $wR_a^M v$. Then since M is
674 memory preserving, v can be extended to a Y -run $\{v_i\}_{i=0}^n$ for which $v = v_0$ and $w'R_a^M v_n$.
675 Note that n can be 0. Then $\{(v_i, t)\}_{i=0}^n$ is a Y -run in $M[U]$ for which $(w', s')R^{M[U]}(v_n, t)$.

676 Suppose instead that s is not a past state. Then $w = w'$, $sR_Y^U s'$, and both $sR_a^U t$ and
677 $wR_a^M v$. Then t can be end-extended to a \underline{Y} -run $\{t_i\}_{i=0}^n$ for which $t = t_0$ and $s'R_a^U t_n$. Note that
678 n can be 0. Because U is past-preserving and non-past-branching, U is path preserving, and
679 hence $\{(v, t_i)\}_{i=0}^n$ is a Y -run in $M[U]$. As $w = w'$, $wR_a^M v$, and $s'R_a^U t_n$, $(w', s')R^{M[U]}(v, t_n)$.

680 **6. S5 Preservation:** To see that $R_a^{M[U]}$ is reflexive, consider a world $(w, s) \in W^{M[U]}$.
681 As R_a^M and R_a^U are reflexive, $wR_a^M w$ and $sR_a^U s$. Hence $(w, s)R_a^{M[U]}(w, s)$.

682 To see that $R_a^{M[U]}$ is symmetric, suppose that $(w, s)R_a^{M[U]}(w', s')$. Then $wR_a^M w'$ and
683 $sR_a^U s'$. By symmetry of $wR_a^M w$ and $sR_a^U s$, $w'R_a^M w$ and $s'R_a^U s$, and hence $(w', s')R^{M[U]}(w, s)$.

684 To see that $R_a^{M[U]}$ is transitive, suppose that $(w, s)R_a^{M[U]}(w', s')$ and $(w', s')R_a^{M[U]}(w'', s'')$.
685 Then $wR_a^M w'$ and $w'R_a^M w''$, and by R_a^M transitivity, $wR_a^M w''$. Similarly $sR_a^U s'$ and $s'R_a^U s''$,
686 and by R_a^U transitivity, $sR_a^U s''$. Hence $(w, s)R_a^{M[U]}(w'', s'')$. \square

5 Embedding Standard DEL

In this section, we show that standard (Temporal) Dynamic Epistemic Logic, which contains neither \underline{Y} - nor Y -arrows, can be embedded in our framework in a natural way. This provides clear connections between our work and the work in [8, 12, 15, 16, 20] on (Temporal) Dynamic Epistemic Logic, which will be described at the end of this section.

Definition 5.1 (Standard). Choose $T \in \{Y, \underline{Y}\}$. To say that a Kripke frame F for $A \cup \{Y, T\}$ is *standard* means that for each $s \in W^F$, we have $R_Y^F(s) = R_{\underline{Y}}^F(s) = \emptyset$. To say that a Kripke model or an L -update frame is *standard* means that the Kripke frame underlying that model or L -update frame is standard. To say that a pointed Kripke model or a pointed L -update frame is *standard* means that the Kripke model or L -update frame making up the first component of the pair is standard.

Definition 5.2 (L_{TDEL} ; [8]). L_{TDEL} is the *Language of Temporal Dynamic Epistemic Logic*. The L_{TDEL} -formulas are the formulas that may be formed by the grammar obtained from that in Definition 2.5 by adding the following formula-formation rule: if φ is an L_{TDEL} -formula and (U, s) is a standard pointed L -update frame with $\emptyset \neq L \subseteq L_{\text{TDEL}}$, then $[U, s]\varphi$ is an L_{TDEL} -formula. L_{TDEL} consists of the L_{TDEL} -formulas along with the L -update frames for which $\emptyset \neq L \subseteq L_{\text{TDEL}}$.

Notation 5.3 (Sequences). Let τ be a finite possibly empty sequence. We write $\tau \cdot x$ to denote the sequence obtained from τ by adding x at the end. The operator “ \cdot ” associates left, so $\tau \cdot x \cdot x'$ means $(\tau \cdot x) \cdot x'$. We write ϵ for the empty sequence. The number of elements in τ is denoted by $|\tau|$. If τ is nonempty, then $\text{last}(\tau)$ denotes the last member of the sequence τ .

Definition 5.4 (Adapted from [9, 12, 15, 16, 20]). A *run* is a nonempty finite sequence $\{M_i\}_{i=0}^n$ of Kripke models satisfying the property that for each $i \in \mathbb{N}$ with $i > 0$ and each $w \in W^{M_i}$, we have that w is of the form (w', s) for some $w' \in W^{M_{i-1}}$. A *pointed run* is a pair $(r \cdot M, w)$ consisting of a run $r \cdot M$ and a world $w \in W^M$; the world w is called the *point* of $(r \cdot M, w)$. A *standard run* is a run whose constituent pointed Kripke models are all standard. A *standard pointed run* is a pointed run (r, w) whose constituent run r is standard. An *L event-run* is a finite possibly empty sequence of pointed L -update frames. A *standard L event-run* is an event run whose constituent pointed L -update frames are all standard.

Notation 5.5 (Projection Functions). We define projection functions π_1 and π_2 on pairs in the usual way; that is, $\pi_1(a, b) := a$ and $\pi_2(a, b) := b$. For each $i \in \mathbb{N}$ and each $j \in \{1, 2\}$, we define the function π_j^i according to the following induction: $\pi_j^0(w) = w$ and $\pi_j^{i+1}(w) = \pi_j(\pi_j^i(w))$. We often write π in place of π_1 and π^i in place of π_j^i .

Definition 5.6 (L_{TDEL} -Truth; [8, 15, 16]). We define a notion of truth for L_{TDEL} -formulas at standard pointed runs by an induction on the construction of L_{TDEL} -formulas.

- $r, w \not\models_{L_{\text{TDEL}}} \perp$ and $r, w \models_{L_{\text{TDEL}}} \top$.
- $r \cdot M, w \models_{L_{\text{TDEL}}} p_k$ means that $w \in V^M(p_k)$.

- 726 • $r, w \models_{L_{\text{TDEL}}} \varphi \star \psi$ means that $r, w \models_{L_{\text{TDEL}}} \varphi$ star $r, w \models_{L_{\text{TDEL}}} \psi$, where “star” is to be
727 replaced by the (mathematical) English reading for the binary Boolean connective \star .
- 728 • $r, w \models_{L_{\text{TDEL}}} \neg\varphi$ means that $r, w \not\models_{L_{\text{TDEL}}} \varphi$.
- 729 • For $a \in A$: $r \cdot M, w \models_{L_{\text{TDEL}}} [a]\varphi$ means that $r \cdot M, x \models_{L_{\text{TDEL}}} \varphi$ for each $x \in R_a^M(w)$.
- 730 • $r \cdot M, w \models_{L_{\text{TDEL}}} [Y]\varphi$ means that if $r \neq \epsilon$, then $r, \pi(w) \models_{L_{\text{TDEL}}} \varphi$.
- 731 • $r \cdot M, w \models_{L_{\text{TDEL}}} [U, s]\varphi$ means that if we have $r \cdot M, w \models_{L_{\text{TDEL}}} \mathbf{p}^U(s)$, then, letting
732 $r' := r \cdot M$, it follows that $r' \cdot r'[U], (w, s) \models_{L_{\text{TDEL}}} \varphi$, where $r'[U]$ is the standard Kripke
733 model defined as follows.

$$W^{r'[U]} \quad := \quad \{(x, t) \in W^M \times W^U : r' \cdot M, x \models_{L_{\text{TDEL}}} \mathbf{p}^U(t)\}$$

for $a \in A$,

$$R_a^{r'[U]}(x, t) \quad := \quad \{(y, u) \in W^{r'[U]} : y \in R_a^M(x) \text{ and } u \in R_a^U(t)\}$$

$$R_Y^{r'[U]}(x, t) \quad := \quad \emptyset$$

$$V^{r'[U]}(p_k) \quad := \quad \{(x, t) \in W^{r'[U]} : r' \cdot M, x \models_{L_{\text{TDEL}}} p_k\}$$

734 When it ought not cause confusion, we may omit the subscript “ L_{TDEL} ” in writing $\models_{L_{\text{TDEL}}}$.

735 **Definition 5.7** (Generated Structures). Let (M, w) be a standard pointed Kripke model.

- 736 • If $\sigma = \{(U_i, s_i)\}_{i=1}^n$ is an L_{DETL} event-run, then $(M, w) \ast^{\mathbf{p}} \sigma$, the *pointed Kripke model*
737 *that is point-generated from (M, w) by σ* , is the pointed Kripke model (M_m, w_m) having
738 the largest integer $m \leq n$ subject to the following restrictions: if $(M_0, w_0) = (M, w)$
739 and $(M_j, w_j) = (M, w) \ast^{\mathbf{p}} \sigma_j$ for each $j \in \mathbb{N}$ with $j < m$, then the following hold.

740 – $w_{j+1} = (w_j, s_{j+1})$.

741 – $(M_j, w_j) \models_{L_{\text{DETL}}} \mathbf{p}^{U_{j+1}}(s_{j+1})$ and $M_{j+1} = M_j[U_{j+1}]$.

742 Note: “ $\models_{L_{\text{DETL}}}$ ” and $M_j[U_{j+1}]$ are given by L_{DETL} -truth (Definition 3.2).

- 743 • If $\sigma = \{(U_i, s_i)\}_{i=1}^n$ is a standard L_{TDEL} event-run, then $(M, w) \ast^{\mathbf{s}} \sigma$, the *pointed run*
744 *that is sequence-generated from (M, w) by σ* , is the pointed run $(\{M_i\}_{i=0}^m, w_m)$ having
745 the largest integer $m \leq n$ subject to the following restrictions: if $(M_0, w_0) = (M, w)$
746 and w_j is the point of $(M, w) \ast^{\mathbf{s}} \sigma_j$ for each $j \in \mathbb{N}$ with $j < m$, then the following hold.

747 – $w_{j+1} = (w_j, s_{j+1})$.

748 – $(\{M_i\}_{i=0}^j, w_j) \models_{L_{\text{TDEL}}} \mathbf{p}^{U_{j+1}}(s_{j+1})$ and $M_{j+1} = \{M_i\}_{i=0}^j[U_{j+1}]$.

749 Note: “ $\models_{L_{\text{TDEL}}}$ ” and $\{M_i\}_{i=0}^j[U_{j+1}]$ are given by L_{TDEL} -truth (Definition 5.6).

$$\begin{aligned}
q^{\sharp n} &:= q \text{ for } q \in \{p_k, \perp, \top\} \\
(\varphi \star \psi)^{\sharp n} &:= \varphi^{\sharp n} \star \psi^{\sharp n} \\
(\neg \varphi)^{\sharp n} &:= \neg(\varphi^{\sharp n}) \\
([a]\varphi)^{\sharp n} &:= [a](\varphi^{\sharp n}) \text{ for } a \in A \text{ or } a = Y, n = 0 \\
([Y]\varphi)^{\sharp n} &:= [Y]\varphi^{\sharp(n-1)} \text{ for } n > 0 \\
([U, s]\varphi)^{\sharp n} &:= [U^{\sharp n}, s](\varphi^{\sharp(n+1)}) \\
\\
W^{U^{\sharp n}} &:= U^W \uplus \{b\} \quad (\text{disjoint union})
\end{aligned}$$

$$\begin{aligned}
\text{for } a \in A \cup \{Y, \underline{Y}\}, \quad R_a^{U^{\sharp n}}(s) &:= \begin{cases} R_a^U(s) & \text{if } a \neq \underline{Y} \text{ and } s \neq b, \\ \{b\} & \text{if } a \neq \underline{Y} \text{ and } s = b, \\ \{b\} & \text{if } a = \underline{Y} \text{ and } s \neq b, \\ \emptyset & \text{if } a = \underline{Y} \text{ and } s = b. \end{cases} \\
\\
\mathbf{p}^{U^{\sharp n}}(s) &:= \begin{cases} (\mathbf{p}^U(s))^{\sharp n} \wedge \langle Y \rangle^n [Y] \perp & \text{if } s \neq b, \\ \top & \text{if } s = b. \end{cases}
\end{aligned}$$

Figure 5: Definition of $\sharp n : L_{\text{TDEL}} \rightarrow L_{\text{DETL}}$ for $n \in \mathbb{N}$

750 **Definition 5.8** ($r \downarrow$). Let $(r, w) = (\{M_i\}_{i=0}^n, w)$ be a standard pointed run sequence-generated
751 by a standard L_{TDEL} event-run from a standard pointed Kripke model. We write $(r, w) \downarrow$ to
752 denote the pointed Kripke model (M, w) defined in the following way.

$$\begin{aligned}
W^M &:= \bigcup_{i=0}^n W^{M_i} \\
R_a^M(v) &:= R_a^{M_i}(v) \text{ for } i \in \mathbb{N} \text{ with } v \in W^{M_i} \\
R_Y^M(v) &:= \begin{cases} \{v'\} & \text{if } v = (v', s) \in W^{M_i} \text{ with } i > 0 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

753 **Definition 5.9** ($\sharp n, \sharp$). For $n \in \mathbb{N}$, we define the function $\sharp n : L_{\text{TDEL}} \rightarrow L_{\text{DETL}}$ and
754 in Figure 5. If $\sigma = \{(U_i, s_i)\}_{i=1}^n$ is a standard L_{TDEL} event-run, then we define $\sigma^\sharp :=$
755 $\{(U_i^{\sharp(i-1)}, s_i)\}_{i=1}^n$.

756 **Lemma 5.10.** Let (M, w) be a standard pointed Kripke model and let σ be a standard
757 L_{TDEL} event-run. Then for each $\varphi \in L_{\text{TDEL}}$,

$$(M, w) *^s \sigma \models_{L_{\text{TDEL}}} \varphi \text{ if and only if } (M, w) *^p \sigma^\sharp \models_{L_{\text{DETL}}} \varphi^{\sharp|\sigma|}.$$

758 *Proof.* We prove this by double induction. The inductive hypotheses assume that for any
759 standard pointed Kripke model (M, s) and any standard L_{TDEL} event run σ ,

$$760 \quad (M, w) *^s \sigma \models_{L_{\text{TDEL}}} \varphi \text{ if and only if } (M, w) *^p \sigma^\sharp \models_{L_{\text{DETL}}} \varphi^{\sharp|\sigma|} .$$

761 The outer inductive hypothesis requires that φ have update modal depth k .⁶ The inner
 762 inductive hypothesis requires that φ be a strict subformula of given formula ψ . The base
 763 cases and most inductive cases are straightforward, so we focus our attention on the most
 764 interesting sub-induction cases:

765 Case $\psi = [Y]\varphi$: Given a pointed Kripke model (M, w) and a standard L_{TDEL} event run
 766 σ of length n , let $(M, w) *^{\text{p}} \sigma^{\sharp} = (N, v)$. Then the following are equivalent:

- 767 1. $(M, w) *^{\text{s}} \sigma \models_{L_{\text{TDEL}}} [Y]\varphi$.
- 768 2. $n > 0$ implies $(M, w) *^{\text{s}} \sigma_{n-1} \models_{L_{\text{TDEL}}} \varphi$.⁷
- 769 3. $n > 0$ implies $(M, w) *^{\text{p}} \sigma_{n-1}^{\sharp} \models_{L_{\text{DETL}}} \varphi^{\sharp(n-1)}$.
- 770 4. $n > 0$ implies $(N, (\pi(v), b)) \models_{L_{\text{DETL}}} \varphi^{\sharp(n-1)}$.
- 771 5. $(M, w) *^{\text{p}} \sigma^{\sharp} \models_{L_{\text{DETL}}} ([Y]\varphi)^{\sharp n}$.

772 The equivalence (1) \Leftrightarrow (2) is by definition of $*^{\text{s}}$ (Definition 5.7). The equivalence (2) \Leftrightarrow (3)
 773 uses the inner inductive hypothesis. The equivalence (3) \Leftrightarrow (4) follows from Theorem 4.5
 774 part 1. The equivalence (4) \Leftrightarrow (5) comes from the fact that $\{(\pi(v), b)\} = R_Y^N(v)$ and
 775 $([Y]\varphi)^{\sharp n} = [Y]\varphi^{\sharp(n-1)}$.

776 Case $\psi = [U, s]\varphi$: Suppose ψ have update modal depth $k + 1$. Given a pointed Kripke
 777 model (M, w) and a standard L_{TDEL} event run σ of length n , the following are equivalent:

- 778 1. $(M, w) *^{\text{s}} \sigma \models_{L_{\text{TDEL}}} [U, s]\varphi$.
- 779 2. $(M, w) *^{\text{s}} \sigma \models_{L_{\text{TDEL}}} \mathbf{p}^U(s)$ implies $(M, w) *^{\text{s}} (\sigma \cdot (U, s)) \models_{L_{\text{TDEL}}} \varphi$.
- 780 3. $(M, w) *^{\text{s}} \sigma \models_{L_{\text{TDEL}}} \mathbf{p}^U(s) \wedge \langle Y \rangle^n [Y]\perp$ implies $(M, w) *^{\text{s}} (\sigma \cdot (U, s)) \models_{L_{\text{TDEL}}} \varphi$.
- 781 4. $(M, w) *^{\text{p}} \sigma^{\sharp} \models_{L_{\text{DETL}}} \mathbf{p}^U(s)^{\sharp n} \wedge \langle Y \rangle^n [Y]\perp$ implies $(M, w) *^{\text{p}} (\sigma \cdot (U, s))^{\sharp} \models_{L_{\text{DETL}}} \varphi^{\sharp(n+1)}$.
- 782 5. $(M, w) *^{\text{p}} \sigma^{\sharp} \models_{L_{\text{DETL}}} ([U, s]\varphi)^{\sharp n}$.

783 The equivalence (1) \Leftrightarrow (2) is by definition of L_{TDEL} -truth (Definition 5.6) and $*^{\text{s}}$ (Defi-
 784 nition 5.7). The equivalence (2) \Leftrightarrow (3) holds because σ has length n . The equivalence
 785 (3) \Leftrightarrow (4) uses the outer inductive hypothesis on both the premise and conclusion of the
 786 implication. The applicability of the inductive hypothesis to the premise follows from the
 787 fact that the update-modal depth of $\mathbf{p}^U(s) \wedge \langle Y \rangle^n [Y]\perp$ is strictly less than the update-modal
 788 depth of $[U, s]\varphi$. The equivalence (4) \Leftrightarrow (5) follows from the definitions of L_{DETL} truth
 789 (Definition 3.2) and of $*^{\text{p}}$ (Definition 5.7) and the fact that $([U, s]\varphi)^{\sharp n} = [U^{\sharp n}, s]\varphi^{\sharp(n+1)}$ and
 790 $\mathbf{p}^{U^{\sharp n}}(s) = \mathbf{p}^U(s) \wedge \langle Y \rangle^n [Y]\perp$. \square

791 **Theorem 5.11** (Isomorphism Theorem). Let (M, w) be a standard pointed Kripke model
 792 and let σ be a standard L_{TDEL} event-run. Then $((M, w) *^{\text{s}} \sigma) \downarrow$ and $(M, w) *^{\text{p}} \sigma^{\sharp}$ are isomor-
 793 phic.⁸

⁶The *update-modal depth* $d(\chi)$ of a L_{DETL} -formula χ is defined by the following induction: $d(q) := 0$ for $q \in \{\perp, \top, p_k\}$, $d(\varphi \star \psi) := \max\{d(\varphi), d(\psi)\}$, $d(\neg\varphi) := d(\varphi)$, $d([a]\varphi) := d(\varphi)$, $d([U, s]\varphi) := 1 + \max\{d(U), d(\varphi)\}$, $d(U) := 1 + \max_{s \in W^U} \{d(\mathbf{p}^U(s))\}$.

⁷If $n - 1 = 0$, then $(M, w) *^{\text{s}} \sigma_{n-1} = (M, w)$.

⁸To say that two (pointed) Kripke models are *isomorphic* means that there exists an isomorphism between them. An *isomorphism between Kripke models* M and M' is a bijection $f : W^M \rightarrow W^{M'}$ satisfying each of the following: (i) $v \in V^M(p_k)$ if and only if $f(v) \in V^{M'}(p_k)$ for each $k \in \mathbb{N}$, and (ii) $u \in R_a^M(v)$ if and only if $f(u) \in R_a^{M'}(f(v))$ for each $a \in A \cup \{Y\}$. An *isomorphism between pointed Kripke models* (M, w) and (M', w') is an isomorphism f between M and M' for which $f(w) = w'$. See [5] for more information.

794 *Proof.* Let $\sigma := \{(U_i, s_i)\}_{i=1}^m$ with $m \in \mathbb{N}$ satisfying $m \leq n$; for $k \in \mathbb{N}$ with $k \leq n$, let
795 $(\bar{M}_k, \bar{w}_k) := ((M, w) *^s \sigma_k) \downarrow$ and let $(\hat{M}_k, \hat{w}_k) := (M, w) *^p \sigma_k^\#$. Also, for each $k < n$, let $\bar{M}_{k+1}^{\leq k}$
796 and $\bar{M}_{k+1}^{=k+1}$ be the submodels of \bar{M}_{k+1} consisting respectively of the worlds in $W^{\bar{M}_{k+1}}$ with
797 Y depth at most k and of the worlds with Y depth $k+1$. Similarly, we let $\hat{M}_{k+1}^{\leq k}$ and $\hat{M}_{k+1}^{=k+1}$
798 be the submodel of \bar{M}_{k+1} consisting respectively of the worlds in $W^{\hat{M}_{k+1}}$ with Y depth at
799 most k and of the worlds with Y depth $k+1$. Then for $0 \leq k < n$, we define two functions:

800 Let $\bar{f}_k : W^{\bar{M}_k} \rightarrow W^{\bar{M}_{k+1}^{\leq k}}$ be the identity function.

801 Let $\hat{f}_k : W^{\hat{M}_k} \rightarrow W^{\hat{M}_{k+1}^{\leq k}}$ be defined by $\hat{f}_k(w) = (w, b)$.

802 Note that $(\bar{M}_0, \bar{w}_0) = (\hat{M}_0, \hat{w}_0) = (M, w)$. Let $f_0 : W^{\bar{M}_0} \rightarrow W^{\hat{M}_0}$ be the identity function.
803 Then for $0 \leq k < n$, we define three functions:

804 Let $g_{k+1} : W^{\bar{M}_{k+1}^{\leq k}} \rightarrow W^{\hat{M}_{k+1}^{\leq k}}$ be defined by $g_{k+1}(w) = (\hat{f}_k \circ f_k \circ \bar{f}_k^{-1})(w)$.

805 Let $h_{k+1} : W^{\bar{M}_{k+1}^{=k+1}} \rightarrow W^{\hat{M}_{k+1}^{=k+1}}$ be defined by $h_{k+1}(v, s) = (g_k(v), s)$.

806 Let $f_{k+1} : W^{\bar{M}_{k+1}} \rightarrow W^{\hat{M}_{k+1}}$ be defined by $f_{k+1}(w) := \begin{cases} g_{k+1}(w) & \text{if } w \in W^{\bar{M}_{k+1}^{\leq k}}, \\ h_{k+1}(w) & \text{otherwise.} \end{cases}$

807 Regarding definability, the only issue that is likely to be of concern is the range specification
808 of h_k . We prove by induction on k that $f_k(v) \in W^{\hat{M}_k^{=k}}$ whenever $(v) \in W^{\bar{M}_k^{=k}}$ (note that
809 f_k and h_k coincide for such v). The base case is $k = 0$, which is immediate from definition.
810 To obtain the inductive step, assume the desired result holds for k . Then the following are
811 equivalent.

- 812 1. $(v, s) \in W^{\bar{M}_{k+1}^{=k+1}}$
- 813 2. $v \in W^{\bar{M}_k^{=k}}$ and $(M, \pi_1^k(v)) *^s \tau \models_{L_{\text{TDEL}}} \mathbf{p}^{U_{k+1}}(s)$, where $\tau = \{(U_i, \pi_2(\pi_1^{k-i}(v)))\}_{i=1}^k$.
- 814 3. $f_k(v) \in W^{\hat{M}_k^{=k}}$ and $(M, \pi_1^k(v)) *^p \tau \models_{L_{\text{DEL}}} \mathbf{p}^{U_{k+1}}(s)$, where $\tau = \{U_i, \pi_2(\pi_1^{k-i}(v))\}_{i=1}^k$.
- 815 4. $f_{k+1}(v, s) \in W^{\hat{M}_{k+1}^{=k+1}}$.

816 The equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4) are almost directly from the definitions. Note
817 that f_{k+1} and h_{k+1} coincide here. The equivalence (2) \Leftrightarrow (3) follows from both the inductive
818 hypothesis and from Lemma 5.10.

819 The goal of this proof is to show that f_m is an isomorphism, and we will prove this by
820 induction on f_k . The base case is f_0 , which is the identity function; and as the domain and
821 range have identical structures, the identity function is an isomorphism. We next assume
822 that f_k is an isomorphism, and we wish to show that f_{k+1} is too. We will be aided by the
823 observation that g_{k+1} is already an isomorphism, as it is the composition of three isomorphic
824 functions: \hat{f}_k , which is isomorphic by definition, f_k by the inductive hypothesis, and \bar{f}_k^{-1} by
825 being the identity function.

826 We first prove that f_{k+1} is a bijective function. Since f_{k+1} is the pasting together of g_{k+1}
827 and h_{k+1} , where g_{k+1} and h_{k+1} have disjoint domains and disjoint ranges, the bijectivity
828 of f_{k+1} follows from the bijectivity of both g_{k+1} and h_{k+1} , whose bijectivity we will now
829 justify. The function g_{k+1} is bijective since it is isomorphic. The bijectivity of h_{k+1} is

830 slightly more involved. To see that it is injective, note that h_{k+1} can be extended to an an
831 injective function, namely the product of f_k with the identity on $W^{U_{k+1}}$, which being the
832 product of injective functions is injective. To see that it is surjective, we observe by the
833 inductive hypothesis that f_k is surjective. Thus if we pick a point $(v, s) \in W^{\hat{M}_{k+1}^{=k+1}}$, then
834 $(f_k^{-1}(v), s) \in W^{\hat{M}_{k+1}^{=k+1}}$, and $h_{k+1}(f_k^{-1}(v), s) = (v, s)$. Therefor h_{k+1} is surjective, and thus
835 bijective too.

836 So what remains in showing that f_{k+1} is an isomorphism is to show the following items:

- 837 (i) $v \in V^{\bar{M}_{k+1}}(p_k)$ if and only if $f_{k+1}(v) \in V^{\hat{M}_{k+1}}(p_k)$ for each $k \in \mathbb{N}$,
- 838 (ii) $v' \in R_a^{\bar{M}_{k+1}}(v)$ if and only if $f_{k+1}(v') \in R_a^{\hat{M}_{k+1}}(f_{k+1}(v))$ for each $a \in A$,
- 839 (iii) $v' \in R_Y^{\bar{M}_{k+1}}(v)$ if and only if $f_{k+1}(v') \in R_Y^{\hat{M}_{k+1}}(f_{k+1}(v))$, and
- 840 (iv) $f_{k+1}(\bar{w}_{k+1}) = \hat{w}_{k+1}$.

841 For (i), since g_{k+1} is an isomorphism, it suffices to take $(v, s) \in W^{\bar{M}_{k+1}^{=k+1}}$ and show that
842 $(v, s) \in V^{\bar{M}_{k+1}}(p_k)$ if and only if $f_{k+1}(v, s) \in V^{\hat{M}_{k+1}}(p_k)$. But by the definition of L_{DEL} -truth
843 (Definition 5.6), we have $(v, s) \in V^{\bar{M}_{k+1}}(p_k)$ if and only if $v \in V^{\bar{M}_k}(p_k)$. By the inductive
844 hypothesis, we have $v \in V^{\bar{M}_k}(p_k)$ if and only if $f_k(v) \in V^{\bar{M}_k}(p_k)$. Applying the definition of
845 L_{DEL} -truth (Definition 3.2), we have $f_k(v) \in V^{\bar{M}_k}(p_k)$ if and only if $(f_k(v), s) \in V^{\hat{M}_k}(p_k)$.
846 This completes the proof of (i).

847 For (ii), since g_{k+1} is an isomorphism, it suffices to take $v \in W^{\bar{M}_{k+1}^{=k+1}}$ and show that
848 $v' \in R_a^{\bar{M}_{k+1}}(v)$ if and only if $f_{k+1}(v') \in R_a^{\hat{M}_{k+1}}(f_{k+1}(v))$. Now $v \in W^{\bar{M}_{k+1}^{=k+1}}$ implies $v = (u, s)$,
849 and the fact that U_{k+1} is standard implies that $v' = (u', s')$. Then the following are equivalent.

- 850 1. $(u', s') \in R_a^{\bar{M}_{k+1}}(u, s)$.
- 851 2. $u' \in R_a^{\bar{M}_k}(u)$ and $s' \in R_a^{U_k}(s)$.
- 852 3. $f_k(u') \in R^{\hat{M}_k}(f_k(u))$ and $s' \in R_a^{U_k^\sharp}(s)$.
- 853 4. $(f_k(u'), s') \in R^{\hat{M}_{k+1}}(f_k(u), s)$.

854 The equivalence (2) \Leftrightarrow (3) follows from the inductive hypothesis and the definition of U_k^\sharp .
855 This completes the proof of (ii).

856 For (iii), since g_{k+1} is an isomorphism, it suffices to take $(u, s) \in W^{\bar{M}_{k+1}^{=k+1}}$, and show that
857 $v \in R_Y^{\bar{M}_{k+1}}(u, s)$ if and only if $f_{k+1}(v) \in R_Y^{\hat{M}_{k+1}}(f_{k+1}(u, s))$. The following are equivalent:

- 858 1. $v \in R_Y^{\bar{M}_{k+1}}(u, s)$.
- 859 2. $v = u$.
- 860 3. $(f_k(v), s) = f_{k+1}(u, s)$.
- 861 4. $f_{k+1}(v) \in R_Y^{\hat{M}_{k+1}}(f_{k+1}(u, s))$.

862 The equivalence (1) \Leftrightarrow (2) follows from the definition of $R_Y^{\bar{M}_{k+1}}$. The equivalence (2) \Leftrightarrow (3)
863 makes use of the definition of h_{k+1} and the injectivity of f_k from the inductive hypothesis.

864 The equivalence (3) \Leftrightarrow (4) follows from the fact that $f_{k+1}(v) = R_Y^{\hat{M}_{k+1}}(f_{k+1}(v), s)$ and the
 865 injectivity of $R_Y^{\hat{M}_{k+1}}$. This completes the proof of (iii).

866 For (iv), the Y -depth of \bar{w}_{k+1} in \bar{M}_{k+1} is $k + 1$, from which it follows that $\bar{w}_{k+1} = (w', s)$
 867 for some $w' \in \bar{M}_k$ with Y -depth k . Further, we have that $f_{k+1}(w', s) = (f_k(w'), s) =$
 868 $(\hat{w}_k, s) = \hat{w}_{k+1}$. This completes the proof of (iv), thereby completing our proof that f_{k+1} is
 869 an isomorphism between $(\bar{M}_{k+1}, \bar{w}_{k+1})$ and $(\hat{M}_{k+1}, \hat{w}_{k+1})$ and so completing the proof of the
 870 theorem. \square

871 The Isomorphism Theorem (Theorem 5.11) allows us to view results about Kripke models
 872 that have been sequence-generated by standard L_{TDEL} event-runs as results about (Temporal)
 873 Dynamic Epistemic Logic—and the other way around. In particular, [15, 16] studies certain
 874 structural properties of the forest structure given by a run $(M, w) *^s \sigma$ that has been sequence-
 875 generated from a standard pointed Kripke model (M, w) by a standard L_{ETL} event-run σ . In
 876 [15, 16], the authors define what it means for the run $(M, w) *^s \sigma$ to be *synchronous* (among
 877 other properties) and then show that every run sequence-generated from a standard pointed
 878 Kripke model by a standard L_{ETL} event-run is synchronous.⁹ Our Preservation Theorem
 879 (Theorem 4.6) works together with the Isomorphism Theorem (Theorem 5.11) to provide
 880 a different perspective on this synchronicity result. In particular, our work shows that the
 881 results of [15, 16] can be viewed as a consequence of the structural properties that are present
 882 in an update frame $U^{\#n}$, produced from a standard update frame U , thereby pinpointing the
 883 source of the synchronicity result in the structure of standard update frames themselves.

884 **Corollary 5.12.** Let (M, w) be a standard pointed Kripke model and let σ be a standard
 885 event-run. Then $\text{last}((M, w) *^p \sigma^\#)$ is synchronous and memory-preserving.

886 6 Examples

887 Suppose Passengers a and b are traveling together by train in China. Further, suppose Pas-
 888 senger a understands Mandarin but that Passenger b does not, though Passenger b mistakenly
 889 believes that they are both equally ignorant of the language.

890 6.1 Public Communication

891 First consider the scenario in which an announcement in English (which both passengers
 892 speak) is made over the loudspeaker. This consists of the *public announcement* of a formula
 893 p . This update frame is represented in Figure 6.

894 The past state s will copy the worlds in the original Kripke model, while t represents the
 895 announcement of φ . Notice that when it occurs, all agents believe that it did. Further, the
 896 \underline{Y} arrow from t to s represents the fact that t occurs one time-step after s .

⁹If $(M, w) *^s \sigma$ is a run sequence-generated from a standard pointed Kripke model (M, w) by a standard L_{ETL} event-run σ , then the definition in [15, 16] would have us say that $(M, w) *^s \sigma$ satisfies *synchronicity* if and only if $((M, w) *^s \sigma) \downarrow$ is synchronous (according to our Definition 4.3).

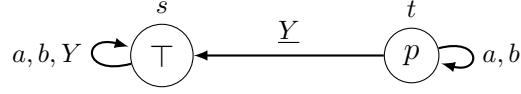


Figure 6: Update frame for the (synchronous) public announcement of φ

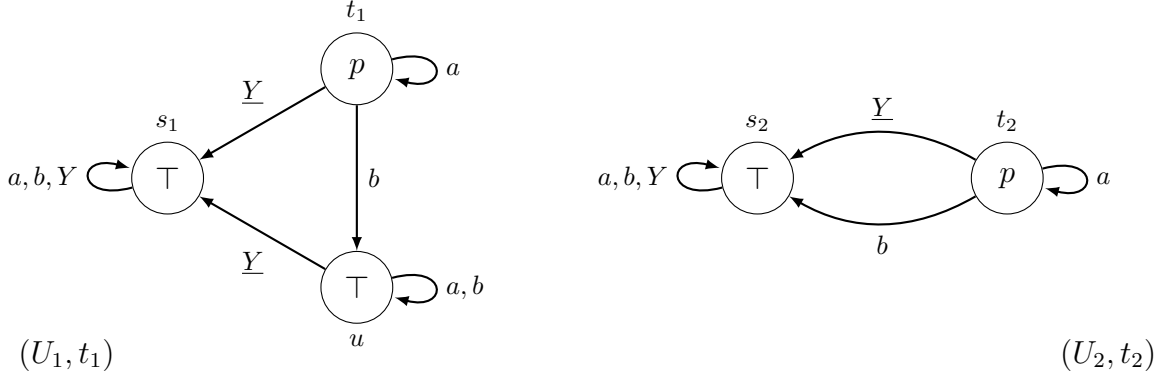


Figure 7: Update frames for the synchronous (left) and asynchronous (right) private announcement of p to a .

897 6.2 Synchronous and Asynchronous Private Communication

898 Now consider two scenarios in which an announcement in Mandarin about a delay in arrival
 899 is made over the loudspeaker.

- 900 1. Passengers a and b are both awake and alert during the announcement.
- 901 2. Passenger a is awake and alert, but Passenger b , who is sleepy, dozes off and sleeps
 902 through the announcement. Waking up a few minutes later without knowing that the
 903 announcement occurred, Passenger b mistakenly thinks that instead of sleeping for a
 904 few minutes, he merely blinked.

905 Taking p to be a propositional letter denoting the statement about late arrival, we repre-
 906 sent the three scenarios in our framework using update frames (U_1, t_1) , (U_2, t_2) , and (U_3, t_3) ,
 907 respectively pictured on the top left, top right, and bottom in Figure 7.

In the first scenario, Passenger b knows that an announcement has taken place, but it provides him with no new information—nor does he believe that a gained any information. In effect, this is a *synchronous private announcement to a* ; after all, both a and b know that an announcement occurred—so the event is *synchronous*—but only a knows the content of the announcement—so the event is *private to a* . In Figure 7, s_1 and u are states in which no new information is conveyed (since \top is always true and thus conveys no new information), while t_1 is a state in which the message p is communicated. Since t_1 and u are each connected to s_1 using a \underline{Y} -arrow, the communications they represent occur one time-step after the communication represented by s_1 . Since s_1 has a reflexive x -arrow for every $x \in \{a, b, Y\}$, state s_1 is in fact a past state; so by the Past State Theorem (Theorem 4.5),

state s_1 represents a description of the situation prior to the update $f_{(U_1, t_1)}$. At state t_1 , Passenger a believes that t_1 represents the only possible communication, while Passenger b believes that u represents the only possible communication. Since both u and t_1 are one time-step after the past state s_1 , the update (U_1, t_1) describes a situation in which it is common knowledge that one time-step occurs. So we see that

$$\models (\neg\langle Y \rangle \top \wedge \neg[b]p) \rightarrow [U_1, t_1]([a]\langle Y \rangle \top \wedge [a]p \wedge [b]\langle Y \rangle \top \wedge \neg[b]p).$$

908 That is, if no event has yet occurred and Passenger b does not believe p , then, after the
 909 occurrence of (U_1, t_1) , Passenger a believes that an event occurred and that p is true, whereas
 910 Passenger b believes that an event occurred but does not believe that p is true.

In contrast, the second scenario is in effect an *asynchronous private announcement to a* . After all, while Passenger a knows that an announcement occurred and she knows its content, Passenger b has two mistaken beliefs: first, that no announcement occurred, and second, that the amount of time between closing and later opening his eyes is essentially negligible. b thus *does not even think it possible* that an event has occurred. Since the announcement results in b having a mistaken belief about the number of events that have occurred, the announcement event is *asynchronous*. At state t_2 in Figure 7, Passenger a knows that p is communicated, but Passenger b mistakenly believes that *no event took place* because the only state he considers possible is the past state s_2 . Accordingly, we see that

$$\models (\neg\langle Y \rangle \top \wedge \neg[b]p) \rightarrow [U_2, t_2]([a]\langle Y \rangle \top \wedge [a]p \wedge \neg[b]\langle Y \rangle \top \wedge \neg[b]p).$$

911 That is, if no event has yet occurred and Passenger b does not believe that p is true, then,
 912 after the occurrence of (U_2, t_2) , Passenger a believes that an event occurred and that p is
 913 true, whereas Passenger b believes neither that an event occurred nor that p is true.

914 These scenarios demonstrate the way in which our framework uses \underline{Y} -arrows to describe
 915 *synchronous* and *asynchronous* private communications. In particular, we see that \underline{Y} -arrows
 916 can be used to describe updates that *need not preserve synchronicity*, as is the case with the
 917 asynchronous private announcement.

918 6.3 Asynchronous Semi-Private Communication

919 Now consider two further scenarios in which passengers observe the passage of stations.

- 920 1. Passenger a is awake, but Passenger b dozes off. While he is asleep, the train passes
 921 a station. Waking up a few minutes later, Passenger b is unsure whether or not any
 922 stations went by. In this case, there is only a single event taking place.
- 923 2. Both passengers are awake and alert as a station passes by. Then, the event described
 924 above occurs, where Passenger a remains awake, but Passenger b sleeps through the
 925 passage of a second station, and wakes up unsure whether any stations went by. In
 926 this case, there are two events taking place.

927 These scenarios are represented as (U_1, t_1) and (U_2, t_2) , respectively in Figure 8.

The first scenario represents an *asynchronous semi-private announcement*. Passenger a knows that an announcement occurred. However, Passenger b does not know whether

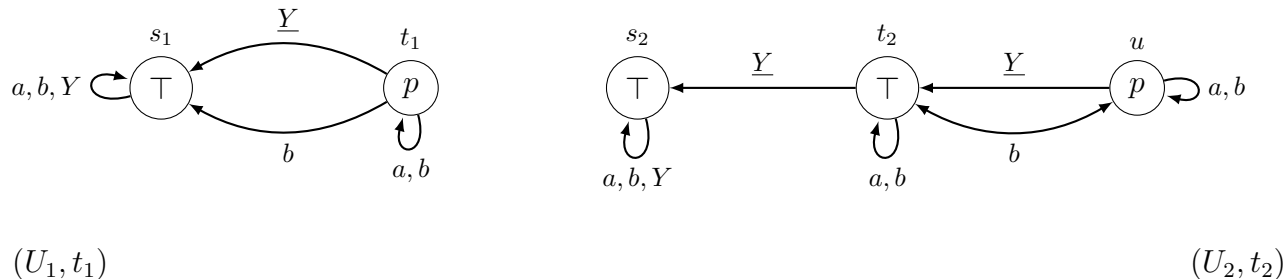


Figure 8: Update frame for the one-step (left) and two-step (right) semi-private announcement of p to a

an announcement occurred, but holds it possible that one did. In this case, b does not have a mistaken belief about the number of events that have occurred, but is uncertain whether there was one event or none. At state t_1 in Figure 8, Passenger a knows that p is communicated, but Passenger b does not know whether the actual world is t_3 or the past state s_3 , in which no event took place. Accordingly, we see that

$$\models (\neg\langle Y \rangle \top \wedge \neg[b]p \wedge \langle b \rangle p) \rightarrow [U_1, t_1]([a]\langle Y \rangle \top \wedge [a]p \wedge \neg[b]\langle Y \rangle \top \wedge \langle b \rangle \langle Y \rangle \top \wedge \neg[b]p \wedge \langle b \rangle p).$$

928 That is, if no event has yet occurred and Passenger b does not believe that p is true, but
 929 holds it possible, then, after the occurrence of (U_1, t_1) , Passenger a believes that an event
 930 occurred and that p is true, whereas b does not believe that an event occurred, but holds it
 931 possible, and does not believe that p is true, but holds it possible. In this case, equivalence
 932 relations are not preserved, because there can be no arrows going from a past state to a state
 933 which is not a past state.

934 The second scenario is also an *asynchronous semi-private announcement*, which, but with
 935 one key technical difference: equivalence relations are preserved. Because t_2 is not a past
 936 state, there can be arrows leaving it. So while the description is in essence the same, (U_2, t_2)
 937 allows us to see that equivalence relations can be preserved if we allow ourselves two events.

938 Now, after this update, we could imagine the train's arrival at the final stop along the
 939 track, just as in the public announcement represented in Figure 6. In that case, both
 940 passengers would know their location. In particular, Passenger b would know that he missed
 941 counting a stop along the way. Thus, we will have $\models K_b(\langle Y \rangle^2 \top \wedge [Y]^3 \perp)$. So Passenger b
 942 knows that two events have taken place.

943 6.4 Non-Memory-Preserving Communication

944 Now consider a different scenario. Suppose Passenger b asks his English-speaking neighbor,
 945 Passenger a , whether the last station was Tianjin. Passenger a gives him an answer, so
 946 b knows either p or $\neg p$. The two then continue on their journey and share a few drinks.
 947 However, having a head neither for city names nor for alcohol, Passenger b forgets what
 948 his neighbor's answer was, though he remembers the exchange. So while he then does not
 949 know whether p , he knows that before drinking he knew whether p was true or false. This
 950 is represented in Figure 9.

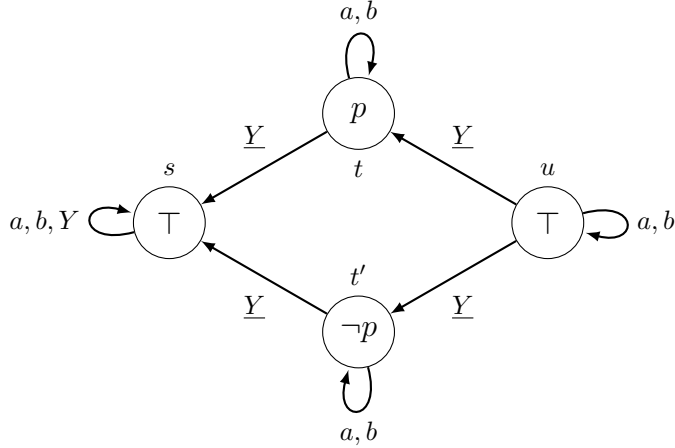


Figure 9: Update frame for memory not being preserved

951 Figure 9 presents an example of an update frame that is not memory-preserving, but
 952 preserves synchronicity and equivalence relations. It is not a standard DEL update frame,
 953 however, because there are \underline{Y} arrows; the update represents a two-step event. However, if
 954 we picture the initial Kripke model as consisting of two worlds, one in which p holds and the
 955 other in which p is false, where the agents have the largest possible epistemic relations, we
 956 will find that they come to know either p or $\neg p$ after the first step, and then fail to know it
 957 after the second step (though they will know that they once knew it). So this is an instance
 958 of agents forgetting a proposition. Other work, particularly [18], has considered more direct
 959 mechanisms for agents’ forgetting the value of a proposition letter.

960 7 The Next Step

961 In its technical essence, this paper is about adding a new type of arrow—the \underline{Y} -arrow—to
 962 update frames and then studying what we can do when the “product update” operation
 963 $M \mapsto M[U]$ due to [3, 4] is extended in such a way that this arrow-type describes a new kind
 964 of operation for use in building an updated Kripke model $M[U]$ from an initial Kripke model
 965 M . In the definition of truth (Definition 3.2), we endowed the \underline{Y} -arrow with the following
 966 meaning: a \underline{Y} -arrow, when present between states $s \in W^U$ and $s' \in W^U$, says that there
 967 is to be a Y -arrow between worlds $(w, s) \in W^{M[U]}$ and $(w', s') \in W^{M[U]}$ if $w = w'$. So the
 968 presence of a \underline{Y} -arrow in an update frame specifies a *sufficient condition* for the creation of
 969 a Y -arrow between two worlds in the updated Kripke model $M[U]$.

970 Arrows in a standard update frame U also specify conditions on the creation of arrows
 971 in the updated Kripke model $M[U]$. In particular, for $a \in A$, if there is an a -arrow between
 972 $s \in W^U$ and $s' \in W^U$, then there is to be an a -arrow between $(w, s) \in W^{M[U]}$ and $(w', s') \in$
 973 $W^{M[U]}$ if there is an a -arrow between $w \in M$ and $w' \in M$. Otherwise, if there is not an
 974 a -arrow between $s \in W^U$ and $s' \in W^U$ —which we will also express by saying, “there is a
 975 *non- a -arrow* between $s \in W^U$ and $s' \in W^U$ ”—then there is to be no a -arrow between worlds
 976 $(w, s) \in W^{M[U]}$ and $(w', s') \in W^{M[U]}$. So we might say that *standard update frames have*

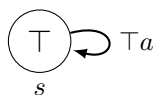
977 *two types of arrows*: the a -arrow and the non- a -arrow. The update frames from this paper
 978 add a third kind of arrow, the \underline{Y} -arrow, into the two-arrow framework of standard Dynamic
 979 Epistemic Logic.

980 When we look at each of the three arrow types studied in our framework—the a -arrow,
 981 the non- a -arrow, and the \underline{Y} -arrow—we see that each type specifies a *logical condition* for the
 982 creation of certain kinds of arrows in the updated Kripke model $M[U]$. An update frame U
 983 combines these logical conditions into a single logical statement that says for each possible
 984 arrow $a \in A \cup \{Y\}$ exactly when it is that there is to be an a -arrow between states (w, s)
 985 and (w', s') in the update Kripke model $M[U]$:¹⁰

- 986 • For $a \in A$, the *a -arrow creation condition* says that there is to be an a -arrow between
 987 $(w, s) \in W^{M[U]}$ and $(w', s') \in W^{M[U]}$ if and only if there is an a -arrow between $w \in W^M$
 988 and $w' \in W^M$ and there is an a -arrow between $s \in W^U$ and $s' \in W^U$.
- 989 • The *Y -arrow creation condition* says that there is to be a Y -arrow between $(w, s) \in$
 990 $W^{M[U]}$ and $(w', s') \in W^{M[U]}$ if and only if (i) there is a Y -arrow between $w \in W^M$
 991 and $w' \in W^M$ and there is a Y -arrow between $s \in W^U$ and $s' \in W^U$, or (ii) $w = w'$
 992 and there is a \underline{Y} -arrow between $s \in W^U$ and $s' \in W^U$.

993 These creation conditions are *fixed logical statements* about the structure of M and of U . But
 994 there are natural logical statements we might like to associate with the creation of certain
 995 kinds of arrows in the updated model $M[U]$.

996 For example, let us use the expression $\top a$ -arrow to refer to new update frame arrow-
 997 type that, when present between states s and s' in U , makes it so that there is an a -arrow
 998 between (w, s) and (w', s') in the updated Kripke model $M[U]$. So if we use the $\top a$ -arrow
 999 in conjunction with the standard a -arrow, then we are led to the following *modified a -arrow*
 1000 *creation condition*: there is to be an a -arrow between $(w, s) \in W^{M[U]}$ and $(w', s') \in W^{M[U]}$ if
 1001 and only if (i) there is an a -arrow between $w \in W^M$ and $w' \in W^M$ and there is an a -arrow
 1002 between $s \in W^U$ and $s' \in W^U$, or (ii) there is a $\top a$ -arrow between $s \in W^U$ and $s' \in W^U$.
 1003 The modified a -arrow creation condition allows us to model certain kinds of “forgetting,” as
 1004 the following small example due to Barteld Kooi (in a private communication) demonstrates:
 1005 the update frame



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 1007 operates on a Kripke model M by producing a Kripke model $M[U]$ in which $R_a^{M[U]}(w, s) =$
 1008 $W^{M[U]}$ for each $(w, s) \in M[U]$. Informally, agent a would “forget” anything he knew in
 1009 M . (Compare this with the work on “introspective forgetting” in [18], which uses valuation
 1010 changes to study a different but related formal notion of forgetting.)

1011 As this example and our work with the \underline{Y} -arrow suggest, we could greatly extend the
 1012 reach of the Dynamic Epistemic Logic approach in a number of interesting and useful ways if

¹⁰Note that this logical statement is always *consistent* by the way the operation $M \mapsto M[U]$ is defined in Definition 3.2. In particular, we have the following two items: (i) for $a \in A$, there is a non- a -arrow between s and s' in U if and only if there is not an a -arrow between s and s' in U ; and (ii) there is a non- Y -arrow between s and s' in U if and only if there is neither a Y -arrow nor a \underline{Y} -arrow between s and s' in U .

1013 we were to devise a language enabling us to explicitly specify the arrow creation conditions
 1014 in the “product update” operation $M \mapsto M[U]$. What we wish to do now is to sketch
 1015 our progress in this direction by introducing our *arrow-precondition language*, which is an
 1016 extension of the following hybrid language.

1017 **Definition 7.1.** Let S be a nonempty finite set of symbols. $\text{HL}(@_z, \forall^1, S)$ consists of the
 1018 formulas formed by the following grammar.

$$\begin{aligned} \varphi ::= & \perp \mid \top \mid p_k \mid z \mid \varphi \star \varphi \mid \neg\varphi \mid [m]\varphi \mid @_z\varphi \mid \forall z.\varphi \\ & k \in \mathbb{N}, \star \in \{\rightarrow, \vee, \wedge, \equiv\}, m \in S \end{aligned}$$

1019 Note that this language contains *no* nominals and has the *one and only* variable z .

1020 **Definition 7.2.** Let S be a nonempty finite set of symbols and let L be a language having
 1021 unary modals $[a]$ for each $m \in S$. A *general L -update frame* is a tuple $(W, \mathbf{p}, \mathbf{a})$ for which

- 1022 • W is a nonempty set whose members will be called *states* (in U),
- 1023 • $\mathbf{p} : W \rightarrow L$ is a function mapping each state $s \in W$ to an L -formula $\mathbf{p}(s)$,
- 1024 • $\mathbf{a} : S \rightarrow (W \times W \rightarrow L)$, which will be called the *arrow-precondition function*, is a
 1025 function mapping each symbol $m \in S$ to a function $\mathbf{a}_m : W \times W \rightarrow L$.

1026 We adopt similar superscript-notation for general L -update frames as we did for update frames
 1027 (Definition 2.4); *pointed general L -update frames* are defined similar to how we defined
 1028 pointed L -update frames (Definition 2.4).

1029 **Definition 7.3.** $L_{\text{DEL}+}$, the *Language of Dynamic Epistemic Logic with Arrow Preconditions*
 1030 *and Yesterday*, is the extension of $\text{HL}(@_z, \forall^1, A \cup \{Y\})$ obtained by adding the following
 1031 formula-formation rule: if φ is an $L_{\text{DEL}+}$ -formula containing no free variables and (U, s) is
 1032 a pointed L -update frame satisfying the property that every L -formula is also an $L_{\text{DEL}+}$ -
 1033 formula containing no free variables, then $[U, s]\varphi$ is an $L_{\text{DEL}+}$ -formula. A (*pointed*) *general*
 1034 *update frame* is a (pointed) general $L_{\text{DEL}+}$ -update frame.

1035 The notion of truth for $L_{\text{DEL}+}$ -formulas is given with respect to pairs $((M, w), g)$ consisting
 1036 of a pointed Kripke model (M, w) and a world $g \in W^M$ that will be called the *designated*
 1037 *world*. The key clauses in the inductive definition of truth for $L_{\text{DEL}+}$ -formulas, most of which
 1038 are standard [13], are as follows.

- 1039 • $(M, w), g \models z$ means that $w = g$.
- 1040 • $(M, w), g \models @_z\varphi$ means that $(M, g), g \models \varphi$.
- 1041 • $(M, w), g \models \forall z.\varphi$ means that $(M, w), g' \models \varphi$ for each $g' \in W^M$.
- 1042 • $(M, w), g \models [a]\varphi$ means that $(M, w'), g \models \varphi$ for each $w' \in R_a^M(w)$.

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- $(M, w), g \models [U, s]\varphi$ means that if $(M, w), g \models \mathbf{p}^U(s)$, then $(M[U], (w, s)), (g, s) \models \varphi$, where $M[U]$ is defined by

$$\begin{aligned} W^{M[U]} &:= \{(v, t) \in W^M \times W^U : (M, v), g \models \mathbf{p}^U(t)\} , \\ R_a^{M[U]}(v, t) &:= \{(v', t') \in W^{M[U]} : (M, v), v' \models \mathbf{a}_a^U(t, t')\} , \text{ and} \\ V^{M[U]}(p_k) &:= \{(v, t) \in W^{M[U]} : (M, v), g \models p_k\} . \end{aligned}$$

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Choosing a L_{DEL^+} -formula φ and a modal $[a]$ with $a \in A \cup \{Y\}$, the following validity explains the role the arrow-precondition function plays in determining how the pointed general update frame (U, s) operates on Kripke models:

$$[U, s][a]\varphi \equiv \mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in W^U} \forall z. (\mathbf{a}_a^U(s, s') \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \quad (2)$$

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It is not too hard to see that general update frames generalize both standard update frames and update frames proper.

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- *Generalizing standard update frames.* Let U be a standard update frame. We define the general update frame $U^+ := (W^U, \mathbf{p}^U, \mathbf{a})$, where \mathbf{a} is defined as follows:

$$\mathbf{a}_a(s, s') := \begin{cases} \langle a \rangle z & \text{if } s' \in R_a^U(s), \\ \perp & \text{if } s' \notin R_a^U(s). \end{cases}$$

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Using the general update frame U^+ in formula (2) yields a formula equivalent to

$$\begin{aligned} [U^+, s][a]\varphi \equiv & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_a^U(s)} \forall z. (\langle a \rangle z \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) \wedge \\ & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in W^U - R_a^U(s)} \forall z. (\perp \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) , \end{aligned}$$

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which is itself equivalent to Axiom $U[a]$ (Figure 1).

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- *Generalizing update frames.* Let U be an update frame. We define the general update frame $U^+ := (W^U, \mathbf{p}^U, \mathbf{a})$, where \mathbf{a} is defined as follows:

$$\mathbf{a}_a(s, s') := \begin{cases} \langle a \rangle z & \text{if } s' \in R_a^U(s) \text{ and } (a \neq Y \text{ or } s' \notin R_Y^U(s)), \\ \perp & \text{if } s' \notin R_a^U(s) \text{ and } (a \neq Y \text{ or } s' \notin R_Y^U(s)), \\ z \vee \langle Y \rangle z & \text{if } a = Y, s' \in R_Y^U(s), \text{ and } s' \in R_Y^U(s), \\ z & \text{if } a = Y, s' \notin R_Y^U(s), \text{ and } s' \in R_Y^U(s). \end{cases}$$

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Using the general update frame U^+ in formula (2) yields a formula that is again equivalent to Axiom $U[a]$ for $a \in A$. But we also have that for $a = Y$, using the general update frame U^+ in formula (2) yields a formula equivalent to

$$\begin{aligned} [U^+, s][Y]\varphi \equiv & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s) - R_Y^U(s)} \forall z. (\langle Y \rangle z \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) \wedge \\ & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in W^U - R_Y^U(s) - R_Y^U(s)} \forall z. (\perp \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) \wedge \\ & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s) \cap R_Y^U(s)} \forall z. (z \vee \langle Y \rangle z \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) \wedge \\ & \left(\mathbf{p}^U(s) \rightarrow \bigwedge_{s' \in R_Y^U(s) - R_Y^U(s)} \forall z. (z \rightarrow @_z(\mathbf{p}^U(s') \rightarrow [U, s']\varphi)) \right) , \end{aligned}$$

1059 which is itself equivalent to Axiom U[Y] (Figure 1).

1060 By allowing us to provide arrow preconditions using conditions expressed explicitly in the
1061 language, $L_{\text{DEL}+}$ vastly expands the domain of applicability of the Dynamic Epistemic Logic
1062 approach. But we have only sketched the details of this line of work, and there is still much
1063 more to be done, including the study of restriction on quantification in $\text{HL}(@_z, \forall^1, S)$ toward
1064 achieving decidability of the satisfiability problem with the least sacrifice in expressivity of
1065 arrow creation conditions for use in formula (2).¹¹ In addition, there a number of new and
1066 interesting arrow types to be identified and studied, including the $\top a$ -type, whose associated
1067 arrow-precondition function sufficiency condition is the formula \top . Such work will contribute
1068 to the expansion in the variety of dynamic multi-agent systems to which we can apply the
1069 Dynamic Epistemic Logic approach.

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¹¹This work may call for clever adaptations of the *standard translation* that would ensure the desired fragment of $\text{HL}(@_z, \forall^1, S)$ is mapped into the *loosely guarded fragment* or *packed fragment* of first-order logic [1, 5, 7, 14].

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