Simple Evidence Elimination in Justification Logic

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Abstract

We present a simple theory for reasoning about evidence and evidence elimination.

1 Introduction

Suppose that the prosecutor presents the jury with exhibit x_1, an audio recording of a boss ordering his subordinate to falsify the accounting ledgers so as to deceive the investors into thinking that his insolvent company is not actually insolvent. Suppose further that the judge provides the jury with oral instructions x_2 stating that the jury may use the following principle in reaching its verdict: "if the boss ordered his subordinate to falsify the ledgers, then the boss is guilty of fraud." Using the principle described by the judge's instructions x_2, the recording x_1 provides the jury with sufficient evidence to find the boss guilty of fraud.

But now suppose that the boss' attorney challenges the authenticity of x_1 (the recording) by presenting further evidence that succeeds in convincing the jury that x_1 (the recording) is not authentic and so should be set aside. This challenge has the effect of *eliminating* the evidence x_1; that is, the challenge makes it so that the jury no longer considers x_1 as evidence that the boss ordered his subordinate to falsify the ledgers. So while the jury still has the judge's instructions x_2 for use in reaching its verdict, it will no longer use x_1 (the recording) as evidence. Assuming that there is no further evidence that the boss ordered his subordinate to falsify the ledgers, the jury will then find the boss not guilty of fraud.

This simplistic example of courtroom evidence presents two important features of evidence. First, evidence is something that can be combined according to logical principles in order to draw conclusions. Second, in drawing conclusions on the basis of evidence, one is sometimes required to set aside (or *eliminate*) certain pieces of evidence and then determine which conclusions can still be drawn using only the evidence that remains.

In this paper, we study a logic for reasoning about these and other issues of evidence. Our logic is an extension of a basic system of *Justification Logic*, a family of logics for reasoning about evidence and justification for rational agents [1, 3, 6, 8]. Justification Logic originated in the proof-theoretic studies of Gödel, who sought an exact provability semantics for the modal logic $\mathsf{S4}$ [5]. Artemov later discovered the *Logic of Proofs* as this long-sought connection between $\mathsf{S4}$ and Gödel's intended $\mathsf{S4}$ provability semantics [2], and a number of authors (including S. Artemov, M. Fitting, R. Iemhoff, N .Krupski, V. Krupski, R. Kuznets, R. Milnikel, B. Renne, and others) have since grown the study of the Logic of Proofs into a broader research project—*Justification Logic*—whose purpose is to investigate a wide-ranging family of logics of evidence and justification for rational agents.

In this paper, we present a system of Justification Logic for reasoning about evidence and evidence elimination. Our theory is called SEE (for Simple Evidence Elimination). We will describe the syntax and semantics of SEE, prove the theory sound and complete with respect to its semantics, and then use our simplistic courtroom evidence example to show how SEE can be used to reason about evidence and evidence elimination.

2 Syntax

The language of SEE allows us to describe propositional truth, the evidence a rational individual holds for a given assertion, and the elimination of such evidence.

Definition 2.1. $\mathfrak{L}(\mathsf{SEE})$ (pronounced "el-ess-e-e"), the *language of Simple Evidence Elimination*, consists of the *terms* t and the *formulas* φ formed by the following grammar.

$$\begin{array}{rcl} t & ::= & c_k \mid x_k \mid t_1 \cdot_\varphi t_2 \mid t_1 + t_2 \mid !t \mid t^{k,\varphi} \\ \varphi & ::= & p_k \mid \bot \mid \top \mid \varphi_1 \star \varphi_2 \mid \neg\varphi \mid t : \varphi \mid [k, \varphi_1]\varphi_2 \mid t :^{k,\varphi_1} \varphi_2 \\ & & k \in \mathbb{N},\ \star \in \{\supset, \wedge, \vee, \equiv\} \end{array}$$

A term of the form c_k is called a *constant* and a term of the form x_k is called a *variable*; the constants and variables make up the *atomic terms*. To say that a term t is *variable-free* means that each non-superscript, non-subscript occurrence of an atomic term in t is a constant. (Examples: $c_0 \cdot_{x_1 : p_5} c_2$ is variable-free, whereas $c_0 \cdot_{x_1 : p_5} x_2$ is not; $c_7^{3,x_8 : p_1}$ is variable-free, whereas $x_7^{3,x_8 : p_1}$ is not.)[1] The p_k's make up the set of *propositional letters*. $\bot$ is the propositional constant for falsity and $\top$ is the propositional constant for truth. Both above and throughout the paper, we use the symbol $\star$ as a metavariable ranging over the binary logical connectives $\supset$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\equiv$ (equivalence). A formula of the form $t : \varphi$ is called an *evidence assertion* and is assigned the informal reading "t is evidence that φ." A formula of the form $[k, \varphi]\psi$ is called an *update assertion*

[1] For Justification Logic aficionados: we will describe the reason we use a subscript formula φ in forming the term $t \cdot_\varphi s$ from terms t and s later in the paper.

and is assigned the informal reading "after [elimination] (k, φ), φ [is true]." Modals of the form $[k, \varphi]$ are called *update modals*. A formula of the form $t :^{x_k,\varphi} \psi$ is called an *elimination assertion* and is assigned the informal reading "[elimination] (k, φ) eliminates evidence t for φ."

Notation 2.2. We let $\mathcal{T}$ denote the set of all terms in $\mathfrak{L}(\mathsf{SEE})$. Whenever it is convenient, we will identify $\mathfrak{L}(\mathsf{SEE})$ with the set of formulas in the language $\mathfrak{L}(\mathsf{SEE})$.

In $\mathfrak{L}(\mathsf{SEE})$, terms play the role of abstract pieces of evidence that may be combined using the term-forming operations given in the grammar of Definition 2.1. The idea is that these term-forming operations represent logical operations of evidence formation. As an example, we will see shortly that $t + s$ is evidence for everything that one or both of t or s evidences. In this way, $t \mapsto t + s$ and $s \mapsto t + s$ each indicate the operation on evidence that takes one piece of evidence and combines it monotonically with another, thereby evidencing all things that were evidenced by one or more of the two constituent pieces.

Formulas of the form $t : \varphi$ express the statement that t is evidence for φ. So we see that the monotonic combination of evidence $t \mapsto t + s$ can be described by the principle $(t : \varphi) \supset (t + s) : \varphi$, which says, "if t is evidence that φ, then $t + s$ is [also] evidence that φ." By writing down a number of principles describing the behavior of the term-forming operations, one can describe a system of evidence satisfying desirable properties.

Since constants will play a special role described later, we will restrict our notion of evidence elimination so as to eliminate evidence assertions of the form $x_k : \varphi$. To make things simple, we will only eliminate one variable at a time, and we will use the formula

$$[k, \varphi]\psi$$

to mean that ψ is true after we eliminate the evidence x_k that φ. This elimination, which we will write as (k, φ), has the effect of making the formula $x_k : \varphi$ false.

The elimination (k, φ) can also have consequences for other pieces of evidence built using x_k. As an example, we saw how the jury's combined evidence (consisting of the judge's instructions combined with the recording) had to be eliminated as a result of eliminating a part of the combination (the recording). So we see that our theory will also need to reason about how the elimination (k, φ) can lead to the elimination of assertions $t : \psi$ in which t is built using x_k. To specify the consequences of the elimination (k, φ) on more complicated pieces of evidence, we will use elimination assertions.

The elimination assertion $t :^{k,\varphi} \psi$ says that the elimination (k, φ) will have the consequence of eliminating $t : \psi$, thereby making it so that $t : \psi$ is false. This allows us to provide schematic descriptions of how an elimination (k, φ) can affect the truth of evidence assertions $t : \psi$ for more complicated terms t.

In the next section, we present the intended semantics for our language $\mathfrak{L}(\mathsf{SEE})$. This semantics describes the conventions we will adopt with regard to evidence behavior in defining our system for reasoning about evidence and evidence elimination.

3 Semantics

Our semantics is based the semantics developed by Fitting [4] and Mkrtychev [7] for the Logic of Proofs. This semantics introduces what we call an *evidence labeling* in order to directly regulate the truth of evidence assertions $t:\varphi$. Placing certain properties of evidence closure on an evidence labeling yields what we call an *evidence function*.

Definition 3.1. An *evidence labeling* is a subset of $\mathcal{T} \times \mathfrak{L}(\mathsf{SEE})$. For a set S of $\mathfrak{L}(\mathsf{SEE})$-formulas, an *$S$-evidence function* is an evidence labeling $\mathcal{A}$ that satisfies each of the following schematic properties.

- *Constant Specification S:* if $k \in \mathbb{N}$ and $\varphi \in S$, then $(c_k, \varphi) \in \mathcal{A}$.
- *Application:* if $(t, \varphi \supset \psi) \in \mathcal{A}$ and $(s, \varphi) \in \mathcal{A}$, then $(t \cdot_\varphi s, \psi) \in \mathcal{A}$.
- *Sum:* if $(t, \varphi) \in \mathcal{A}$ or $(s, \varphi) \in \mathcal{A}$, then $(t + s, \varphi) \in \mathcal{A}$.
- *Checker:* if $(t, \varphi) \in \mathcal{A}$, then $(!t, t:\varphi) \in \mathcal{A}$.
- *Update:* if $(t, \varphi) \in \mathcal{A}$, then $(t^{k,\psi}, [k, \psi]\varphi) \in \mathcal{A}$.

If it is convenient and unlikely to cause confusion, we may drop the prefix "S-" in referring to an S-evidence function.

Remark 3.2. We will use an evidence labeling $\mathcal{A}$ to determine the truth of evidence assertions $t:\varphi$ in the following way: $(t, \varphi) \in \mathcal{A}$ will mean that $t:\varphi$ is true. Under this reading, the defining properties of an S-evidence function give us the following connection between term-forming operations and evidence closure principles.

- Constant Specification S tells us that c_k is evidence for φ whenever φ is in S. Thinking of the set S as a collection of "basic statements" that are to be accepted without detailed justification, this property has us use the constants as evidence for the statements that have been identified as "basic." We will later take S as the set of axioms in our to-be-defined axiomatic theory for simple evidence elimination, thereby identifying the axioms of the theory as the "basic statements" that will be evidenced by a constant.

- Application tell us that $t \cdot_\varphi s$ is evidence for ψ whenever t is evidence for $\varphi \supset \psi$ and s is evidence for φ. Thus $t \cdot_\varphi s$ represents the combination of the evidence t for $\varphi \supset \psi$ with the evidence s for φ so as to evidence ψ according to the rule of *Modus Ponens*:

  $$\frac{\varphi \supset \psi \quad \varphi}{\psi} \ ,$$

  which is read, "from assumptions $\varphi \supset \psi$ and φ, conclude ψ." The subscript φ in $t \cdot_\varphi s$ indicates the important rule φ plays as the antecedent of $\varphi \supset \psi$ in the above application of Modus Ponens.

- Sum tells us that $t + s$ is evidence for everything evidenced by one or more of t and s. So $t + s$ is the monotonic combination of evidence t with evidence s.

- Checker tells us that in case t is evidence for φ, then $!t$ checks that t is evidence for φ. So $!t$ provides a means of verifying an evidence assertion.

- Update tells us that in case t is evidence for φ, then $t^{k,\psi}$ is evidence that φ is true after elimination (k, ψ). To make sense of this, if we think of t as very strong evidence that φ is always true, then we ought to be able to use t in an argument showing that φ is true after elimination (k, ψ) by virtue of the fact that φ is always true. We use the term $t^{k,\psi}$ to represent this argument.

While an evidence labeling (and thus an evidence function) will allow us to determine the truth of evidence assertions $t : \varphi$, we still need a way to determine the truth of propositional letters. This is the purpose of a *valuation*.

Definition 3.3. A *valuation* is a set of propositional letters.

We will use a valuation V to determine propositional truth in the following way: $p_k \in V$ means that p_k is true. This is all we need to determine propositional truth.

Taken together, an evidence labeling $\mathcal{A}$ and a valuation V make up a pair $(\mathcal{A}, V)$ that we call an *evidenced valuation*. (An evidenced valuation whose evidence labeling is in fact an evidence function is what we will call a *model*.) Evidenced valuations provide all the ingredients we need to define a notion of truth for $\mathfrak{L}(\mathsf{SEE})$-formulas.

Definition 3.4. Let S be a set of $\mathfrak{L}(\mathsf{SEE})$-formulas. An *evidenced valuation* is a pair $(\mathcal{A}, V)$ consisting of an evidence labeling $\mathcal{A}$ and a valuation V. An S*-model* is an evidenced valuation $(\mathcal{A}, V)$ satisfying the property that $\mathcal{A}$ is an S-evidence function. If it is convenient and unlikely to cause confusion, we may drop the prefix "S-" in referring to an S-model.

Definition 3.5 (Truth)**.** Let $(\mathcal{A}, V)$ be an evidenced valuation. For an $\mathfrak{L}(\mathsf{SEE})$-formula φ, we write $\mathcal{A}, V \models \varphi$ to mean that φ is *true in* $(\mathcal{A}, V)$, and we write $\mathcal{A}, V \not\models \varphi$ to mean that φ is not true in $(\mathcal{A}, V)$. We define the notion of truth for an $\mathfrak{L}(\mathsf{SEE})$-formula in the evidenced valuation $(\mathcal{A}, V)$ by the following induction on $\mathfrak{L}(\mathsf{SEE})$-formula construction.

- $\mathcal{A}, V \models p_k$ means that $p_k \in V$.

- $\mathcal{A}, V \not\models \bot$ and $\mathcal{A}, V \models \top$.

- $\mathcal{A}, V \models \varphi_1 \star \varphi_2$ means that $\mathcal{A}, V \models \varphi_1$ star $\mathcal{A}, V \models \varphi_2$ for $\star \in \{\supset, \wedge, \vee, \equiv\}$.[2]

- $\mathcal{A}, V \models \neg\varphi$ means that $\mathcal{A}, V \not\models \varphi$.

[2]The word "star" is to be replaced by the English reading for the binary logical connective $\star$; in particular, $\supset$ is read "implies", $\wedge$ is read "and", $\vee$ is read "or", and $\equiv$ is read "if and only if." Note that the connectives $\supset$ and $\equiv$ are to be understood as being defined in the appropriate way in terms of the *material conditional*, which is given by saying that "φ implies ψ" is true exactly when φ is false or ψ is true.

Axiom Scheme

$$\text{EV.} \quad x_k : \varphi$$

Rules

$$\frac{\vdash t : (\psi \supset \chi)}{\vdash (t \cdot_\psi s) : \chi}\ (\text{EAL}) \qquad \frac{\vdash s : \psi}{\vdash (t \cdot_\psi s) : \chi}\ (\text{EAR})$$

$$\frac{\vdash t : \psi \qquad \vdash s : \psi}{\vdash (t + s) : \psi}\ (\text{ES})$$

$$\frac{\vdash t : \psi}{\vdash !t : (t : \psi)}\ (\text{EC})$$

$$\frac{\vdash t : \psi}{\vdash t^{j,\chi} : [j, \chi]\psi}\ (\text{EU})$$

Figure 1: The theory $\mathsf{E}(k, \varphi)$

- $\mathcal{A}, V \models t : \varphi$ means that $(t, \varphi) \in \mathcal{A}$.
- $\mathcal{A}, V \models t :^{k,\varphi} \psi$ means that $\mathsf{E}(k, \varphi) \vdash t : \psi$.

  The theory $\mathsf{E}(k, \varphi)$ is defined in Figure 1. We will write $\mathsf{E}(k, \varphi) \vdash t : \psi$ to mean that the $\mathfrak{L}(\mathsf{SEE})$-formula $t : \psi$ is derivable in $\mathsf{E}(k, \varphi)$, and we will write $\mathsf{E}(k, \varphi) \nvdash t : \psi$ to mean that $t : \psi$ is not derivable in $\mathsf{E}(k, \varphi)$. Our reason for using the theory $\mathsf{E}(k, \varphi)$ will be explained in a moment.

- $\mathcal{A}, V \models [k, \varphi]\psi$ means that $\mathcal{A}[k, \varphi], V \models \psi$, where

$$\mathcal{A}[k, \varphi] := \{(t, \chi) \in \mathcal{A} \mid \mathsf{E}(k, \varphi) \nvdash t : \chi\} \ .$$

The definition of truth (Definition 3.5) identifies the truth of evidence assertions $t : \varphi$ in an evidenced valuation $(\mathcal{A}, V)$ with the contents of the evidence labeling $\mathcal{A}$, in the sense that $\mathcal{A}, V \models t : \varphi$ if and only if $(t, \varphi) \in \mathcal{A}$. Thus we see that if an evidence valuation $(\mathcal{A}, V)$ happens to be a model (Definition 3.4), which means that $\mathcal{A}$ is an evidence function (Definition 3.1), then the truth of evidence assertions $t : \varphi$ is regulated in a way that respects the intended meanings of the term-forming operations (described in Remark 3.2).

Before we describe the other key clauses within our definition of truth, let us first take a moment to recall and then flesh out the motivating ideas behind our notion of evidence elimination. First, we represent eliminations using a pair (k, φ) consisting of a natural number $k \in \mathbb{N}$ and a formula $\varphi \in \mathfrak{L}(\mathsf{SEE})$. An elimination (k, φ) is to eliminate certain evidence assertions $t : \psi$, in the sense that the occurrence of the elimination (k, φ) will make it the case that $t : \psi$ is false for certain evidence assertions $t : \psi$. As for determining the evidence assertions $t:\psi$ that ought to be eliminated under the elimination (k,φ), we use the following principles.

- *Elimination Base:* (k,φ) eliminates $x_k:\varphi$.
- *Elimination Triggers.* For each of the evidence function properties (Definition 3.1) other than Constant Specification S, use the inverse of the property to trigger eliminations of evidence assertions $t:\psi$ based on the evidence assertions that have already been eliminated. (Note: in reading the inverse of an evidence function property from Definition 3.1 for the purpose of this principle, we interpret the negation of an assertion $(t,\psi)\in\mathcal{A}$ as saying, "elimination (k,φ) eliminates $t:\psi$.") Written in detail, this principle specifies the following elimination triggers.
  - *Inverse Application Trigger:* if (k,φ) eliminates $t:(\psi\supset\chi)$ or (k,φ) eliminates $s:\psi$, then (k,φ) also eliminates $(t\cdot_\psi s):\chi$.
  - *Inverse Sum Trigger:* if (k,φ) eliminates $t:\psi$ and (k,φ) eliminates $s:\psi$, then (k,φ) also eliminates $(t+s):\psi$.
  - *Inverse Checker Trigger:* if (k,φ) eliminates $t:\psi$, then (k,φ) also eliminates $!t:(t:\psi)$.
  - *Inverse Update Trigger:* if (k,φ) eliminates $t:\psi$, then (k,φ) also eliminates $t^{j,\chi}:[j,\chi]\psi$.

The idea behind the elimination principles is that the evidence function properties describe logical principles of evidence closure that intuitively connect the veracity of one or more evidence assertions $s_1:\chi_1$ and $s_2:\chi_2$ with the veracity of an evidence assertion $t(s_1,s_2):\chi$ whose evidence $t(s_1,s_2)$ is built out of s_1 and s_2 using one of the term-forming operations (Definition 2.1). In essence, the term-forming operation that allows us to construct $t(s_1,s_2)$ out of the terms s_1 and s_2 is to be identified with a certain logical principle for constructing the more complicated piece of evidence $t(s_1,s_2)$ for χ out of the simpler pieces of evidence s_1 (for χ_1) and s_2 (for χ_2) according to our description in Remark 3.2. So when we eliminate one or more of the evidence assertions $s_1:\chi_1$ and $s_2:\chi_2$, thereby undermining the veracity of each assertion that we eliminate, we may end up undermining the veracity of the assertion $t(s_1,s_2):\chi$ because the veracity of this assertion intuitively depends on the veracity of one or more of $s_1:\chi_1$ and $s_2:\chi_2$. Whether this happens depends on whether the elimination (k,φ) has falsified the antecedent of the evidence function property governing the term-forming operation that lets us form $t(s_1,s_2)$ from s_1 and s_2. Illustrative example: if the elimination (k,φ) eliminates $t:\psi$ and $s:\psi$, then this has the effect of falsifying the antecedent of the Sum property ("if $t:\psi$ or $s:\psi$, then $(t+s):\psi$"). But falsifying the antecedent of the Sum property has the intuitive effect of undermining the veracity of the evidence assertion $(t+s):\psi$ because $t+s$ is supposed to evidence all those things that are evidenced by one or more of t and s (see Remark 3.2). Therefore, if (k,φ) eliminates $t:\psi$ and $s:\psi$, then (k,φ) should also eliminate $(t+s):\psi$. (Note that the statement in the previous sentence is just the inverse of the Sum property, where we use the reading of inverses specified above in the description of the Elimination Triggers principle.) As this illustrative example has shown, the inverse of an evidence function property tells us when the elimination of certain evidence assertions intuitively ought to trigger the elimination of another evidence assertion. So this is why we specified the Elimination Triggers property as we did above.

As an example of how an elimination affects the truth of evidence assertions, let us name a few of the evidence assertions $s:\chi$ that are to be eliminated by an occurrence of the elimination $(1,\varphi)$. First, the elimination $(1,\varphi)$ will obviously eliminate $x_1:\varphi$ due to the principle of Elimination Base. But since $(1,\varphi)$ eliminates $x_1:\varphi$, the Inverse Application Trigger says that $(1,\varphi)$ must also eliminate $(t\cdot_\varphi x_1):\psi$. So the elimination $(1,\varphi)$ eliminates both $x_1:\varphi$ and $(t\cdot_\varphi x_1):\psi$. But these eliminations trigger further eliminations, including the elimination of $(x_1+x_1):\varphi$ (by the Inverse Sum Trigger), the elimination of $!(t\cdot_\varphi x_1):((t\cdot_\varphi x_1):\psi)$ (by the Inverse Checker Trigger), and the elimination of $x_1^{2,\psi}:[2,\psi]\varphi$ (by the Inverse Update Trigger), along with many other eliminations. This is how the elimination $(1,\varphi)$ brings about the elimination of a wide variety of evidence assertions $s:\chi$.

We now examine the way in which the definition of truth handles elimination assertions $t:^{k,\varphi}\psi$. Our intention is that $t:^{k,\varphi}\psi$ is true in an evidenced valuation $(\mathcal{A},V)$ if and only if the elimination (k,φ) eliminates $t:\psi$. (It is in this way that elimination assertions allow us to describe the effects of the elimination (k,φ) within our formal language.) So we see that the truth of an elimination assertion $t:^{k,\varphi}\psi$ is identified with the action of the elimination (k,φ) on the evidence assertion $t:\psi$. Since we have said that we want this action to follow the logical closure principles described by the principles of Elimination Base and Elimination Triggers, whether the action of the elimination (k,φ) affects the evidence assertion $t:\psi$ is a question of logical consequence and it is not hard to see that this notion of logical consequence is encapsulated by the simple axiomatic theory $\mathsf{E}(k,\varphi)$ in Figure 1. This is the reason why the truth of an elimination assertion $t:^{k,\varphi}\psi$ in an evidenced valuation has been identified with the derivability of $t:\psi$ in the theory $\mathsf{E}(k,\varphi)$.

Some readers may find this reliance on the axiomatic theory $\mathsf{E}(k,\varphi)$ within our definition of truth a bit strange because we are connecting the notion of derivability in the axiomatic theory $\mathsf{E}(k,\varphi)$, a syntactic notion, with our definition of truth, a semantic notion. Unfortunately, some such dependence is unavoidable in our framework because we insist that the action of the elimination (k,φ) on evidence assertions ensures that whenever evidence assertions $s_1:\psi_1$ and $s_2:\psi_2$ are eliminated, then so are the evidence assertions $t(s_1,s_2):\psi$ whose evidence $t(s_1,s_2)$ has its veracity intuitively dependent on the veracity of the evidence of one or more of s_1 (for ψ_1) and s_2 (for ψ_2) in the way we described above. Since this notion of dependence is of an essentially logical nature, the notion of truth must somehow utilize the notion of logical consequence encapsulated by the theory $\mathsf{E}(k,\varphi)$. But note that such a notion of logical dependence is just what we want: think of our example of simplistic courtroom evidence, where the elimination of evidence x_1 (the recording of the boss ordering his subordinate to falsify the ledgers) is to bring about the elimination of the logical combination of evidence obtained by joining evidence x_1 (the recording) with evidence x_2 (the judge's instructions that the boss is guilty of fraud if he orders his subordinate to falsify the ledgers).

The reader who is still suspicious of the above connection between our notion of truth and $\mathsf{E}(k,\varphi)$-derivability will hopefully find some comfort in the fact that the theory $\mathsf{E}(k,\varphi)$ is extremely simple and well-behaved. In particular, notice that the conclusion of each rule produces an evidence assertion $t'{:}\varphi'$ with a term t' that is more complex (contains more symbols) than any term t occurring in a hypothesis $t{:}\varphi$ of the rule. Further, there is a one-to-one correspondence between the syntactic term-forming operations (from Definition 2.1) and the rules of $\mathsf{E}(k,\varphi)$. Also, the lone Axiom EV of $\mathsf{E}(k,\varphi)$ pertains to atomic terms (in fact, to the single variable x_k for a fixed $k \in \mathbb{N}$). It is thus not difficult to see that this theory is decidable.[3]

Finally, let us look at the definition of truth for update assertions $[k,\varphi]\psi$. Our intention is to have $[k,\varphi]\psi$ true in an evidence valuation $(\mathcal{A},V)$ if and only if ψ is true after we alter the evidence labeling $\mathcal{A}$ according to the action of the elimination (k,φ) on evidence assertions. But since we used the theory $\mathsf{E}(k,\varphi)$ to characterize those evidenced assertions $t{:}\chi$ that are to be eliminated, in the sense that (k,φ) eliminates $t{:}\chi$ if and only if $\mathsf{E}(k,\varphi) \vdash t{:}\chi$, then we alter $\mathcal{A}$ by removing all term-formula pairs $(t,\chi) \in \mathcal{A}$ such that $\mathsf{E}(k,\varphi) \vdash t{:}\chi$. Thus we see that in defining $\mathcal{A}[k,\varphi]$ by setting

$$\mathcal{A}[k,\varphi] := \{(t,\chi) \in \mathcal{A} \mid \mathsf{E}(k,\varphi) \nvdash t{:}\chi\} \ ,$$

the evidence labeling $\mathcal{A}[k,\varphi]$ ensures that the evidence assertions $t{:}\chi$ true in $(\mathcal{A}[k,\varphi],V)$, the evidence labeling that obtains after the occurrence of the elimination (k,φ), are just those evidence assertions $t{:}\chi$ that were true in $(\mathcal{A},V)$ before the occurrence of (k,φ) and were also left intact by the action of (k,φ) on evidence assertions.

The notion of formula validity in which we will be interested is given relative to a set S of "basic assertions" that are to be justified using a constant.

Definition 3.6 (Validity)**.** Let S be a set of $\mathfrak{L}(\mathsf{SEE})$-formulas and φ be an $\mathfrak{L}(\mathsf{SEE})$-formula. To say that φ is *S-valid*, written $S \models \varphi$, means that $\mathcal{A},V \models \varphi$ for each S-model $(\mathcal{A},V)$. We write $S \not\models \varphi$ to mean that φ is not S-valid.

Since our notion of formula validity is given relative a set S of "basic assertions," it will be important to see that the semantic elimination operation

$$(\mathcal{A},V) \mapsto (\mathcal{A}[k,\varphi],V)$$

from Definition 3.5 maps S-models to S-models. Said informally, we wish to see that this operation preserves the property of being an S-model.

Lemma 3.7 (S-Model Preservation)**.** Let S be a set of $\mathfrak{L}(\mathsf{SEE})$-formulas. If $(\mathcal{A},V)$ is an S-model, then $(\mathcal{A}[k,\varphi],V)$ is also an S-model.

[3] In particular, determining whether $\mathsf{E}(k,\varphi) \vdash t{:}\psi$ is $O(2^{|t|})$, where $|t|$ is equal to the number of occurrences of term-forming operations that were used in constructing t out of variables and constants according to the grammar in Definition 2.1.

Proof. It suffices for us to show that $\mathcal{A}[k,\varphi]$ is an S-evidence function under the assumption that $\mathcal{A}$ is an S-evidence function. According to the definition of S-evidence functions (Definition 3.1), $\mathcal{A}[k,\varphi]$ is an S-evidence function if and only if it satisfies each of Constant Specification S, Application, Sum, Checker, and Update.

Let us check that $\mathcal{A}[k,\varphi]$ satisfies Constant Specification S. We observe that the axiomatics of $\mathsf{E}(k,\varphi)$ in Figure 1 ensures that $\mathsf{E}(k,\varphi)\nvdash c_j:\varphi$ for every $j\in\mathbb{N}$. Applying the definition of $\mathcal{A}[k,\varphi]$ (Definition 3.5), it follows that $(c_j,\psi)\in\mathcal{A}$ if and only if $(c_j,\psi)\in\mathcal{A}[k,\varphi]$ for each $j\in\mathbb{N}$. But then the fact that $\mathcal{A}$ satisfies Constant Specification S implies that $\mathcal{A}[k,\varphi]$ satisfies Constant Specification S.

Let us check that $\mathcal{A}[k,\varphi]$ satisfies Application. We observe that the axiomatics of $\mathsf{E}(k,\varphi)$ in Figure 1 ensures that $\mathsf{E}(k,\varphi)\nvdash(t\cdot_\psi s):\chi$ if and only if $\mathsf{E}(k,\varphi)\nvdash t:(\psi\supset\chi)$ and $\mathsf{E}(k,\varphi)\nvdash s:\psi$. Applying the definition of $\mathcal{A}[k,\varphi]$ (Definition 3.5), we have that $(t,\psi\supset\chi)\in\mathcal{A}[k,\varphi]$ and $(s,\psi)\in\mathcal{A}[k,\varphi]$ together imply that $(t,\psi\supset\chi)\in\mathcal{A}$, $\mathsf{E}(k,\varphi)\nvdash t:(\psi\supset\chi)$, $(s,\psi)\in\mathcal{A}$, and $\mathsf{E}(k,\varphi)\nvdash s:\psi$. Since $\mathcal{A}$ satisfies Application, $(t,\psi\supset\chi)\in\mathcal{A}$ and $(s,\psi)\in\mathcal{A}$ together imply that $(t\cdot_\psi s,\chi)\in\mathcal{A}$. And the result from the second sentence of this paragraph shows that $\mathsf{E}(k,\varphi)\nvdash t:(\psi\supset\chi)$ and $\mathsf{E}(k,\varphi)\nvdash s:\psi$ together imply that $\mathsf{E}(k,\varphi)\nvdash(t\cdot_\psi s):\chi$. Applying again the definition of $\mathcal{A}[k,\varphi]$, we have shown that $(t,\psi\supset\chi)\in\mathcal{A}[k,\varphi]$ and $(s,\psi)\in\mathcal{A}[k,\varphi]$ together imply that $(t\cdot_\psi s,\chi)\in\mathcal{A}[k,\varphi]$. It follows that $\mathcal{A}[k,\varphi]$ satisfies Application.

The argument that $\mathcal{A}[k,\varphi]$ satisfies each of Sum, Checker, and Update is shown similarly. Conclusion: $\mathcal{A}[k,\varphi]$ is an S-evidence function. □

4 Axiomatics

We are now in a position to describe the axiomatics of our theory of Simple Evidence Elimination, SEE.

Definition 4.1. SEE (pronounced "ess-e-e"), the *Theory of Simple Evidence Elimination*, is defined in Figure 2. For each $\mathfrak{L}(\mathsf{SEE})$-formula φ, we write $\mathsf{SEE}\vdash\varphi$ to mean that φ is derivable in SEE and we write $\mathsf{SEE}\nvdash\varphi$ to mean that φ is not derivable in SEE.

SEE an extension of a basic theory of Justification Logic.[4] Like other Justification Logics, SEE satisfies Artemov's *Internalization Theorem*, which provides a sense in which the structure of terms can mirror reasoning within the theory.

Theorem 4.2 (Artemov's Internalization Theorem; [2])**.** $\mathsf{SEE}\vdash\varphi$ implies $\mathsf{SEE}\vdash t:\varphi$ for a variable-free term $t\in\mathcal{T}$.

[4]This theory was called J4 in [8], though we note that the languages of J4 and SEE vary slightly. In particular, while the language of SEE, $\mathfrak{L}(\mathsf{SEE})$, uses a subscript formula φ in forming the term $t\cdot_\varphi s$ from terms t and s, the language of J4 forms the term $t\cdot s$ from terms t and s without a subscript formula. We require such a subscript formula in $\mathfrak{L}(\mathsf{SEE})$ in order to be able to express that the formula $(t\cdot_\psi s):^{k,\varphi}\chi$ is equivalent to some Boolean combination of formulas of the form $t:^{k,\varphi}\chi_1$ and $s:^{k,\varphi}\chi_2$ for appropriate χ_1 and χ_2. The equivalence we want is Axiom X3 of SEE (Figure 2).

Classical Logic and Evidence

CL.	Axiom schemes for classical propositional logic
E1.	$\big(t : (\varphi \supset \psi)\big) \supset \big((s : \varphi) \supset (t \cdot_{\varphi} s) : \psi\big)$
E2.	$(t : \varphi) \supset (t + s) : \varphi$
	$(s : \varphi) \supset (t + s) : \varphi$
E3.	$(t : \varphi) \supset\, !t : (t : \varphi)$
E4.	$(t : \varphi) \supset t^{k,\psi} : [k, \psi]\varphi$

Update and Elimination

U1.	$[k, \varphi]q$	$\equiv$	q
U2.	$[k, \varphi](\psi \star \chi)$	$\equiv$	$[k, \varphi]\psi \star [k, \varphi]\chi$
U3.	$[k, \varphi]\neg\psi$	$\equiv$	$\neg[k, \varphi]\psi$
U4.	$[k, \varphi](t : \psi)$	$\equiv$	$(t : \psi) \wedge \neg(t :^{k,\varphi} \psi)$
U5.	$[k, \varphi](t :^{j,\chi} \psi)$	$\equiv$	$(t :^{j,\chi} \psi)$
X1.	$(c_j :^{k,\varphi} \psi)$	$\equiv$	$\bot$
X2.	$(x_j :^{k,\varphi} \psi)$	$\equiv$	$\begin{cases} \top & \text{if } (j, \psi) = (k, \varphi) \\ \bot & \text{otherwise} \end{cases}$
X3.	$\big((t \cdot_{\psi} s) :^{k,\varphi} \chi\big)$	$\equiv$	$\big(t :^{k,\varphi} (\psi \supset \chi)\big) \vee (s :^{k,\varphi} \psi)$
X4.	$\big((t + s) :^{k,\varphi} \psi\big)$	$\equiv$	$(t :^{k,\varphi} \psi) \wedge (s :^{k,\varphi} \psi)$
X5.	$\big(!t :^{k,\varphi} \theta\big)$	$\equiv$	$\begin{cases} t :^{k,\varphi} \chi & \text{if } \theta = (t : \chi) \\ \bot & \text{otherwise} \end{cases}$
X6.	$\big(t^{j,\chi} :^{k,\varphi} \theta\big)$	$\equiv$	$\begin{cases} t :^{k,\varphi} \psi & \text{if } \theta = [j, \chi]\psi \\ \bot & \text{otherwise} \end{cases}$

Note: $q \in \{p_k, \bot, \top\}$ and $\star \in \{\supset, \wedge, \vee, \equiv\}$.

Rules

$$\frac{k \in \mathbb{N} \quad \varphi \text{ an axiom}}{\vdash c_k : \varphi}\ (\text{CN})$$

$$\frac{\vdash \varphi \supset \psi \quad \vdash \varphi}{\vdash \psi}\ (\text{MP}) \qquad \frac{k \in \mathbb{N} \quad \vdash \varphi}{\vdash [k, \psi]\varphi}\ (\text{UN})$$

Figure 2: The theory SEE

Proof. By induction on the length of the SEE-derivation of φ. In the base case, φ is an axiom, and it follows from Rule CN that $\mathsf{SEE} \vdash c_0 : \varphi$. Taking $t := c_0$, a variable-free term, the result follows. In the induction step, φ follows by a rule of inference. We consider each rule of inference in turn.

- Induction Case: $\varphi = (c_k : \psi)$ follows from ψ by Rule CN.

  Take $t := !c_k$, a variable-free term. We observe that $\mathsf{SEE} \vdash t : \varphi$ by Axiom E3 and Rule MP.

- Induction Case: φ follows from $\psi \supset \varphi$ and ψ by Rule MP.

  By the induction hypothesis, there are variable-free terms s_1 and s_2 such that $\mathsf{SEE} \vdash s_1 : (\psi \supset \varphi)$ and $\mathsf{SEE} \vdash s_2 : \psi$. Applying Axiom E1 and Rule MP, it follows that $\mathsf{SEE} \vdash (s_1 \cdot_\psi s_2) : \varphi$. Taking $t := s_1 \cdot_\psi s_2$, we observe that t is variable-free.

- Induction Case: $\varphi = [k, \psi]\chi$ follows from χ by Rule UN.

  By the induction hypothesis, there is a variable-free term s such that $\mathsf{SEE} \vdash s : \chi$. It follows by Axiom E4 and Rule MP that $\mathsf{SEE} \vdash s^{k,\psi} : [k, \psi]\chi$. Taking $t := s^{k,\psi}$, we observe that t is variable-free. □

The Internalization Theorem provides a sense in which rational agents can formulate specific arguments describing the process of logical deduction; that is, in deducing φ using a specific SEE-deduction, the term t yielded by the proof of the Internalization Theorem provides an explicit description of the step-by-step reasoning that took place in the deduction. This bolsters the sense in which terms serve as pieces of evidence in theories of Justification Logic.

Our Soundness Theorem says that if we take S to be the set of SEE-axioms, thereby equating these axioms with the "basic statements" that are to be justified by a constant, then all SEE-theorems are S-valid.

Theorem 4.3 (Soundness)**.** Let S be the set of SEE-axioms. $\mathsf{SEE} \vdash \varphi$ implies $S \models \varphi$.

Proof. We show by induction on the length of derivation in SEE that each SEE-theorem is S-valid. In the base case of this induction, we must show that each SEE-axiom is S-valid.

- Base Case: Axiom CL is S-valid.

  This follows from the usual truth-table arguments for classical propositional logic.

- Base Case: Axioms E1–E4 are S-valid.

  Let $(\mathcal{A}, V)$ be an S-model. That E1 is true in $(\mathcal{A}, V)$ follows from the definition of truth (Definition 3.5) and the fact that $\mathcal{A}$ satisfies Application. Similarly, E2 is true in $(\mathcal{A}, V)$ because $\mathcal{A}$ satisfies Sum, E3 is true in $(\mathcal{A}, V)$ because $\mathcal{A}$ satisfies Checker, and E4 is true in $(\mathcal{A}, V)$ because $\mathcal{A}$ satisfies Update. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that E1–E4 are each S-valid.

- Base Case: Axioms U1–U5 are S-valid.

  That each of Axioms U1–U5 is S-valid follows directly from the definition of truth (Definition 3.5). The most interesting case is Axiom U4, so let us write out the argument for this axiom as a paradigmatic example for the others.

  Let $(\mathcal{A}, V)$ be an S-model. To have $\mathcal{A}, V \models [k, \varphi](t : \psi)$ means that $\mathcal{A}[k, \varphi], V \models t : \psi$, which itself means that $(t, \psi) \in \mathcal{A}[k, \varphi]$. By the definition of $\mathcal{A}[k, \varphi]$ (Definition 3.5), $(t, \psi) \in \mathcal{A}[k, \varphi]$ is equivalent to $(t, \psi) \in \mathcal{A}$ and $\mathsf{E}(k, \varphi) \nvdash t : \psi$. By the definition of truth, the latter conjunction is itself equivalent to the statement that $\mathcal{A}, V \models t : \psi$ and $\mathcal{A}, V \not\models t :^{k,\varphi} \psi$. Again applying the definition of truth, the latter conjunction is equivalent to $\mathcal{A}, V \models (t : \psi) \wedge \neg(t :^{k,\varphi} \psi)$. We therefore have shown that $\mathcal{A}, V \models [k, \varphi](t : \psi)$ is equivalent to $\mathcal{A}, V \models (t : \psi) \wedge \neg(t :^{k,\varphi} \psi)$, and so it follows by the definition of truth that U4 is true in $(\mathcal{A}, V)$. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that U4 is S-valid.

- Base Case: Axiom X1 is S-valid.

  Let $(\mathcal{A}, V)$ be an S-model. By the definition of truth, $\mathcal{A}, V \models (c_j :^{k,\varphi} \psi) \equiv \bot$ is equivalent to $\mathsf{E}(k, \varphi) \nvdash c_j : \psi$. By an examination of the axiomatics of $\mathsf{E}(k, \varphi)$ from Figure 1, the latter is simply true. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that X1 is S-valid.

- Base Case: Axiom X2 is S-valid.

  Let $(\mathcal{A}, V)$ be an S-model. By the definition of truth, $\mathcal{A}, V \models (x_k :^{k,\varphi} \varphi) \equiv \top$ is equivalent to $\mathsf{E}(k, \varphi) \vdash x_k : \varphi$. By an examination of the axiomatics of $\mathsf{E}(k, \varphi)$ from Figure 1, the latter is simply true.

  Also by the definition of truth, $\mathcal{A}, V \models (x_j :^{k,\varphi} \psi) \equiv \bot$ for $(j, \psi) \neq (k, \varphi)$ is equivalent to $\mathsf{E}(k, \varphi) \nvdash x_j : \psi$. By an examination of the axiomatics of $\mathsf{E}(k, \varphi)$ from Figure 1, it follows from our assumption $(j, \psi) \neq (k, \varphi)$ that $\mathsf{E}(k, \varphi) \nvdash x_j : \psi$ is simply true.

  Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that X2 is S-valid.

- Base Case: Axiom X3 is S-valid.

  Let $(\mathcal{A}, V)$ be an S-model. By the definition of truth, $\mathcal{A}, V \models (t \cdot_\psi s) :^{k,\varphi} \chi$ is equivalent to $\mathsf{E}(k, \varphi) \vdash (t \cdot_\psi s) : \chi$. By an examination of the axiomatics of $\mathsf{E}(k, \varphi)$ from Figure 1, the latter is equivalent to the statement that $\mathsf{E}(k, \varphi) \vdash t : (\psi \supset \chi)$ or $\mathsf{E}(k, \varphi) \vdash s : \psi$. But the latter disjunction is what it means to say that $\mathcal{A}, V \models \big(t :^{k,\varphi} (\psi \supset \chi)\big) \vee (s :^{k,\varphi} \psi)$. We therefore have shown that $\mathcal{A}, V \models (t \cdot_\psi s) :^{k,\varphi} \chi$ is equivalent to $\mathcal{A}, V \models \big(t :^{k,\varphi} (\psi \supset \chi)\big) \vee (s :^{k,\varphi} \psi)$, and so it follows from the definition of truth that X3 is true in $(\mathcal{A}, V)$. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that X3 is S-valid.

- Base Case: Axioms X4–X6 are S-valid.

  These are shown by arguments similar to the above argument for Axiom X3.

$$\begin{array}{lcl} d(q) & := & 0 \\ d(\varphi \star \psi) & := & \max\{d(\varphi), d(\psi)\} \\ d(\neg\varphi) & := & d(\varphi) \\ d(t:\varphi) & := & 0 \\ d(t:^{k,\varphi}\psi) & := & 0 \\ d\big([k,\varphi]\psi\big) & := & 1 + d(\psi) \end{array}$$

Note: $q \in \{p_k, \bot, \top\}$ and $\star \in \{\supset, \wedge, \vee, \equiv\}$.

Figure 3: Definition of a function $d : \mathfrak{L}(\mathsf{SEE}) \to \mathbb{N}$.

This completes the base cases of the induction. For the induction cases, we are to show that the SEE-rules preserve S-validity. We consider each rule in turn.

- Induction Case: $c_k : \varphi$ was derived from φ using Rule CN.

  Let $(\mathcal{A}, V)$ be an S-model. φ is an SEE-axiom and therefore $\varphi \in S$. It follows that $(c_k, \varphi) \in \mathcal{A}$ by the fact that $\mathcal{A}$ satisfies Constant Specification S. But then $\mathcal{A}, V \models c_k : \varphi$. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that $c_k : \varphi$ is S-valid.

- Induction Case: ψ was derived from $\varphi \supset \psi$ and φ using Rule MP.

  By the induction hypothesis, each of $\varphi \supset \psi$ and φ is S-valid. Let $(\mathcal{A}, V)$ be an S-model. It follows from the S-validity of $\varphi \supset \psi$ and φ that $\mathcal{A}, V \models (\varphi \supset \psi) \wedge \varphi$ and thus that $\mathcal{A}, V \models \psi$. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that ψ is S-valid.

- Induction Case: $[k, \psi]\varphi$ was derived from φ using Rule UN.

  By the induction hypothesis, φ is S-valid. Let $(\mathcal{A}, V)$ be an S-model. It follows from the S-Model Preservation Lemma (Lemma 3.7) that $(\mathcal{A}[k, \psi], V)$ is an S-model. Applying the S-validity of φ, we then have that $\mathcal{A}[k, \psi], V \models \varphi$. But the latter is what it means to have $\mathcal{A}, V \models [k, \psi]\varphi$. Since $(\mathcal{A}, V)$ was an arbitrarily chosen S-model, we have shown that $[k, \psi]\varphi$ is S-valid. □

The converse of the Soundness Theorem (Theorem 4.3) is the Completeness Theorem. To prove the Completeness Theorem, we introduce a notion of *depth* for $\mathfrak{L}(\mathsf{SEE})$-formulas that will come up later.

Definition 4.4 ($\mathfrak{L}(\mathsf{SEE})$-Depth)**.** The $\mathfrak{L}(\mathsf{SEE})$-*depth function* is a function $d : \mathfrak{L}(\mathsf{SEE}) \to \mathbb{N}$ that maps each formula $\varphi \in \mathfrak{L}(\mathsf{SEE})$ to a natural number $d(\varphi)$ according to the definition in Figure 3. We call $d(\varphi)$ the *depth* of φ.

$$
\begin{array}{lcl}
q^\circ & := & q \\
(\varphi \star \psi)^\circ & := & \varphi^\circ \star \psi^\circ \\
(\neg\varphi)^\circ & := & \neg\varphi^\circ \\
(t:\varphi)^\circ & := & t:\varphi \\
(t:^{k,\varphi}\psi)^\circ & := & t:^{k,\varphi}\psi \\
\big([k,\varphi]q\big)^\circ & := & q \\
\big([k,\varphi](\psi\star\chi)\big)^\circ & := & \big([k,\varphi]\psi\big)^\circ \star \big([k,\varphi]\chi\big)^\circ \\
\big([k,\varphi]\neg\psi\big)^\circ & := & \neg\big([k,\varphi]\psi\big)^\circ \\
\big([k,\varphi](t:\psi)\big)^\circ & := & (t:\psi) \wedge \neg(t:^{k,\varphi}\psi) \\
\big([k,\varphi](t:^{j,\chi}\psi)\big)^\circ & := & t:^{j,\chi}\psi \\
\big([k,\varphi][j,\psi]\chi\big)^\circ & := & \big([k,\varphi]\big([j,\psi]\chi\big)^\circ\big)^\circ
\end{array}
$$

Note: $q \in \{p_k, \bot, \top\}$ and $\star \in \{\supset, \wedge, \vee, \equiv\}$.

Figure 4: Definition of a function $\circ : \mathfrak{L}(\mathsf{SEE}) \to \mathfrak{L}(\mathsf{SEE})$.

As it turns out, each $\mathfrak{L}(\mathsf{SEE})$-formula φ is provably equivalent in SEE to an $\mathfrak{L}(\mathsf{SEE})$-formula φ° with $d(\varphi^\circ) = 0$, which says that φ° does not contain occurrences of update modals within the scope of a term.[5] The formula φ°, called the *reduction* of φ, is defined as follows.

Definition 4.5. The $\mathfrak{L}(\mathsf{SEE})$-*reduction function* is a function $\circ : \mathfrak{L}(\mathsf{SEE}) \to \mathfrak{L}(\mathsf{SEE})$ that maps each formula $\varphi \in \mathfrak{L}(\mathsf{SEE})$ to the formula $\varphi^\circ \in \mathfrak{L}(\mathsf{SEE})$ according to the definition in Figure 4. We call φ° the *reduction* of φ.

Lemma 4.6 (Reduction Lemma)**.** $d(\varphi^\circ) = 0$ and $\mathsf{SEE} \vdash \varphi \equiv \varphi^\circ$.

Proof. By an induction on the construction of φ. Abbreviations: $\vdash \gamma$ abbreviates $\mathsf{SEE} \vdash \gamma$, "SEE" abbreviates "reasoning in SEE", "IH" abbreviates "induction hypothesis."

- Base Case: $\varphi = q$ for $q \in \{p_k, \bot, \top\}$.

$$
\begin{array}{rcll}
d(q^\circ) & = & d(q) & \text{Figure 4} \\
& = & 0 & \text{Figure 3}
\end{array}
$$

$$
\begin{array}{lll}
1. & \vdash q \equiv q & \text{by } \mathsf{SEE} \\
2. & \vdash q \equiv q^\circ & \text{by 1, Figure 4}
\end{array}
$$

[5]To say that an $\mathfrak{L}(\mathsf{SEE})$-formula θ contains a piece of syntax *within the scope of a term* means that there is a subformula $t:\psi$ of θ such that there is an occurrence of the piece of syntax in ψ.

- Induction Case: $\varphi = (\psi \star \chi)$.

$$\begin{array}{lll} & d((\psi \star \chi)^\circ) & \\ = & d(\psi^\circ \star \chi^\circ) & \text{by Figure 4} \\ = & \max\{d(\psi^\circ), d(\chi^\circ)\} & \text{by Figure 3} \\ = & 0 & \text{by IH} \end{array}$$

$$\begin{array}{lll} 1. & \vdash \psi \equiv \psi^\circ & \text{by IH} \\ 2. & \vdash \chi \equiv \chi^\circ & \text{by IH} \\ 3. & \vdash (\psi \star \chi) \equiv (\psi^\circ \star \chi^\circ) & \text{by 1, 2, } \mathsf{SEE} \\ 4. & \vdash (\psi \star \chi) \equiv (\psi \star \chi)^\circ & \text{by 3, Figure 4} \end{array}$$

- Induction Case: $\varphi = \neg\psi$.

$$\begin{array}{lll} & d((\neg\psi)^\circ) & \\ = & d(\neg\psi^\circ) & \text{by Figure 4} \\ = & d(\psi^\circ) & \text{by Figure 3} \\ = & 0 & \text{by IH} \end{array}$$

$$\begin{array}{lll} 1. & \vdash \psi \equiv \psi^\circ & \text{by IH} \\ 2. & \vdash \neg\psi \equiv \neg\psi^\circ & \text{by 1, } \mathsf{SEE} \\ 3. & \vdash \neg\psi \equiv (\neg\psi)^\circ & \text{by 2, Figure 4} \end{array}$$

- Induction Case: $\varphi = (t : \psi)$.

$$\begin{array}{lll} & d((t : \psi)^\circ) & \\ = & d(t : \psi) & \text{by Figure 4} \\ = & 0 & \text{by Figure 3} \end{array}$$

$$\begin{array}{lll} 1. & \vdash (t : \psi) \equiv (t : \psi) & \text{by } \mathsf{SEE} \\ 2. & \vdash (t : \psi) \equiv (t : \psi)^\circ & \text{by 1, Figure 4} \end{array}$$

- Induction Case: $\varphi = (t :^{k,\psi} \chi)$.

  Similar to the previous induction case ($\varphi = t : \psi$).

- Induction Case: $\varphi = [k, \psi]\theta$.

  By a sub-induction on the depth $d(\theta)$ of θ with a sub-sub-induction on the construction of θ. Abbreviations: "SIH" abbreviates "sub-induction hypothesis" and "SSIH" abbreviates "sub-sub-induction hypothesis." A reference to an SEE axiom indicates that the result is by reasoning in SEE that makes crucial use of the axiom in question.

  - Sub-Base Case: $d(\theta) = 0$; Sub-Sub-Base Case: $\theta = q$, where $q \in \{p_k, \bot, \top\}$.

$$\begin{array}{llll} d(([k, \psi]q)^\circ) & = & d(q) & \text{by Figure 4} \\ & = & 0 & \text{by Figure 3} \end{array}$$

1. $\vdash [k,\psi]q \equiv q$ by Axiom U1
2. $\vdash [k,\psi]q \equiv ([k,\psi]q)^\circ$ by 1, Figure 4

- Sub-Base Case: $d(\theta) = 0$; Sub-Sub-Induction Case: $\theta = (\chi \star \omega)$.

$$\begin{aligned} & d(([k,\psi](\chi \star \omega))^\circ) \\ = \ & d(([k,\psi]\chi)^\circ \star ([k,\psi]\omega)^\circ) && \text{by Figure 4} \\ = \ & \max\{d(([k,\psi]\chi)^\circ), d(([k,\psi]\omega)^\circ)\} && \text{by Figure 3} \\ = \ & 0 && \text{by SSIH} \end{aligned}$$

1. $\vdash [k,\psi]\chi \equiv ([k,\psi]\chi)^\circ$ by SSIH
2. $\vdash [k,\psi]\omega \equiv ([k,\psi]\omega)^\circ$ by SSIH
3. $\vdash ([k,\psi]\chi \star [k,\psi]\omega) \equiv ([k,\psi]\chi)^\circ \star ([k,\psi]\omega)^\circ$ by 1, 2, SEE
4. $\vdash ([k,\psi]\chi \star [k,\psi]\omega) \equiv ([k,\psi](\chi \star \omega))^\circ$ by 3, Figure 4
5. $\vdash [k,\psi](\chi \star \omega) \equiv ([k,\psi](\chi \star \omega))^\circ$ by 4, Axiom U2

- Sub-Base Case: $d(\theta) = 0$; Sub-Sub-Induction Case: $\theta = \neg\chi$.

$$\begin{aligned} d(([k,\psi]\neg\chi)^\circ) &= d(\neg([k,\psi]\chi)^\circ) && \text{by Figure 4} \\ &= d(([k,\psi]\chi)^\circ) && \text{by Figure 3} \\ &= 0 && \text{by SSIH} \end{aligned}$$

1. $\vdash [k,\psi]\chi \equiv ([k,\psi]\chi)^\circ$ by SSIH
2. $\vdash \neg[k,\psi]\chi \equiv \neg([k,\psi]\chi)^\circ$ by 1, SEE
3. $\vdash \neg[k,\psi]\chi \equiv ([k,\psi]\neg\chi)^\circ$ by 2, Figure 4
4. $\vdash [k,\psi]\neg\chi \equiv ([k,\psi]\neg\chi)^\circ$ by 3, Axiom U3

- Sub-Base Case: $d(\theta) = 0$; Sub-Sub-Induction Case: $\theta = (t : \chi)$.

$$\begin{aligned} & d(([k,\psi](t : \chi))^\circ) \\ = \ & d((t : \chi) \wedge \neg(t :^{k,\psi} \chi)) && \text{by Figure 4} \\ = \ & 0 && \text{by Figure 3} \end{aligned}$$

1. $\vdash [k,\psi](t : \chi) \equiv (t : \chi) \wedge \neg(t :^{k,\psi} \chi)$ by Axiom U4
2. $\vdash [k,\psi](t : \chi) \equiv ([k,\psi](t : \chi))^\circ$ by Figure 4

- Sub-Base Case: $d(\theta) = 0$; Sub-Sub-Induction Case: $\theta = (t :^{j,\chi} \omega)$.

$$\begin{aligned} & d(([k,\psi](t :^{j,\chi} \omega))^\circ) \\ = \ & d(t :^{j,\chi} \omega) && \text{by Figure 4} \\ = \ & 0 && \text{by Figure 3} \end{aligned}$$

1. $\vdash [k,\psi](t :^{j,\chi} \omega) \equiv (t :^{j,\chi} \omega)$ by Axiom U5
2. $\vdash [k,\psi](t :^{j,\chi} \omega) \equiv ([k,\psi](t :^{j,\chi} \omega))^\circ$ by Figure 4

– Sub-Induction Case: $d(\theta) > 0$; Sub-Sub-Base and Sub-Sub-Induction Case: $\theta = [j,\chi]\omega$.

By the SIH, we have that $d(([j,\chi]\omega)^\circ) = 0$. It therefore follows that

$$d([k,\psi]([j,\chi]\omega)^\circ) = 1$$

by Figure 3. Applying the fact that $d([k,\psi][j,\chi]\omega) \geq 2$, we have shown that the SIH also applies to the formula $[k,\psi]([j,\chi]\omega)^\circ$, which gives us the following.

$$\begin{array}{lll} & d(([k,\psi][j,\chi]\omega)^\circ) & \\ = & d(([k,\psi]([j,\chi]\omega)^\circ)^\circ) & \text{by Figure 4} \\ = & 0 & \text{by SIH} \end{array}$$

$$\begin{array}{lll} 1. & \vdash [j,\chi]\omega \equiv ([j,\chi]\omega)^\circ & \text{by SIH} \\ 2. & \vdash [k,\psi][j,\chi]\omega \equiv [k,\psi]([j,\chi]\omega)^\circ & \text{by 1, } \mathsf{SEE} \\ 3. & \vdash [k,\psi]([j,\chi]\omega)^\circ \equiv ([k,\psi]([j,\chi]\omega)^\circ)^\circ & \text{by SIH} \\ 4. & \vdash [k,\psi][j,\chi]\omega \equiv ([k,\psi]([j,\chi]\omega)^\circ)^\circ & \text{by 2, 3, } \mathsf{SEE} \\ 5. & \vdash [k,\psi][j,\chi]\omega \equiv ([k,\psi][j,\chi]\omega)^\circ & \text{by 4, Figure 4} \end{array}$$

– Sub-Induction Case: $d(\theta) > 0$, Sub-Sub-Induction Cases $\theta = (\chi \star \omega)$, $\theta = \neg\chi$, $\theta = (t : \chi)$, and $\theta = (t :^{j,\chi} \omega)$ are handled as in the corresponding sub-sub-induction cases of Sub-Induction Case $d(\theta) = 0$. (See above.) □

In addition to the Reduction Lemma (Lemma 4.6), we will need one more lemma to facilitate our proof of the forthcoming Completeness Theorem. This lemma is as follows.

Lemma 4.7. $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$ if and only if $\mathsf{E}(k,\varphi) \vdash t : \psi$.

Proof. Let S be the set of SEE-axioms. It is not difficult to see that $(\mathcal{T} \times \mathfrak{L}(\mathsf{SEE}), \emptyset)$ is an S-model.[6] It therefore follows by soundness (Theorem 4.3) and the definition of validity (Definition 3.6) that $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$ implies $\mathcal{T} \times \mathfrak{L}(\mathsf{SEE}), \emptyset \models t :^{k,\varphi} \psi$. Applying the definition of truth, the latter implies $\mathsf{E}(k,\varphi) \vdash t : \psi$. So we see that $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$ implies $\mathsf{E}(k,\varphi) \vdash t : \psi$. To prove the converse of this implication, we argue by induction on the length of derivation in $\mathsf{E}(k,\varphi)$ that $\mathsf{E}(k,\varphi) \vdash t : \psi$ implies $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$. (The axiomatics of $\mathsf{E}(k,\varphi)$ are defined in Figure 1, and the axiomatics of SEE are defined in Figure 2.)

- Base Case: $\mathsf{E}(k,\varphi) \vdash x_k : \varphi$ by Axiom EV of $\mathsf{E}(k,\varphi)$.

  We have $\mathsf{SEE} \vdash x_k :^{k,\varphi} \varphi$ by Axiom X2 of SEE.

- Induction Case: $\mathsf{E}(k,\varphi) \vdash (t \cdot_\psi s) : \chi$ using Rule EAL of $\mathsf{E}(k,\varphi)$.

  By the induction hypothesis, $\mathsf{SEE} \vdash t :^{k,\varphi} (\psi \supset \chi)$. Reasoning in SEE, it follows that $\mathsf{SEE} \vdash \big(t :^{k,\varphi} (\psi \supset \chi)\big) \vee (s :^{k,\varphi} \psi)$. Applying Axiom X3 of SEE, the result follows.

[6] It is easy to see that $\mathcal{A}^* := \mathcal{T} \times \mathfrak{L}(\mathsf{SEE})$ satisfies each of the defining properties of S-evidence functions (Definition 3.1) by the fact that $(t,\psi) \in \mathcal{A}^*$ holds for every $(t,\psi) \in \mathcal{T} \times \mathfrak{L}(\mathsf{SEE})$.

- Induction Case: $\mathsf{E}(k,\varphi) \vdash (t \cdot_\psi s) : \chi$ using Rule EAR of $\mathsf{E}(k,\varphi)$.

  Similar to the previous induction case.

- Induction Case: $\mathsf{E}(k,\varphi) \vdash (t + s) : \psi$ using Rule ES of $\mathsf{E}(k,\varphi)$.

  By the induction hypothesis, $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$ and $\mathsf{SEE} \vdash s :^{k,\varphi} \psi$. Reasoning in SEE, it follows that $\mathsf{SEE} \vdash (t :^{k,\varphi} \psi) \wedge (s :^{k,\varphi} \psi)$. Applying Axiom X4 of SEE, the result follows.

- Induction Case: $\mathsf{E}(k,\varphi) \vdash !t : (t : \psi)$ using Rule EC of $\mathsf{E}(k,\varphi)$.

  By the induction hypothesis, $\mathsf{SEE} \vdash t :^{k,\varphi} \psi$. Applying Axiom X5 of SEE, the result follows.

- Induction Case: $\mathsf{E}(k,\varphi) \vdash t^{j,\chi} : [j,\chi]\psi$ using Rule EU of $\mathsf{E}(k,\varphi)$.

  Similar to the previous induction case, except that we use Axiom X6 of SEE. □

Our Completeness Theorem says that if we take S to be the set of SEE-axioms, thereby equating these axioms with the "basic statements" that are to be justified by a constant, then the S-valid $\mathfrak{L}(\mathsf{SEE})$-formulas are SEE-theorems.

Theorem 4.8 (Completeness)**.** Let S be the set of SEE-axioms. $S \models \varphi$ implies $\mathsf{SEE} \vdash \varphi$.

Proof. Let us make a few definitions leading up to a canonical model argument. A *conjunction* of a finite set of $\mathfrak{L}(\mathsf{SEE})$-formulas is the conjunction whose conjuncts are the members of the finite set. To say that a formula φ *implies* $\bot$ means that $\mathsf{SEE} \vdash \varphi \supset \bot$. To say that a set of $\mathfrak{L}(\mathsf{SEE})$-formulas is *consistent* means that no conjunction of a finite subset implies $\bot$. To say that a set of $\mathfrak{L}(\mathsf{SEE})$-formulas is *inconsistent* means that the set is not consistent. To say that a set of $\mathfrak{L}(\mathsf{SEE})$-formulas is *maximal consistent* means that the set is consistent and adding any $\mathfrak{L}(\mathsf{SEE})$-formula not already in the set would produce an inconsistent set. Any consistent set of $\mathfrak{L}(\mathsf{SEE})$-formulas may be extended to a maximal consistent set of $\mathfrak{L}(\mathsf{SEE})$-formulas using a Lindenbaum Argument.

Assume that $\mathsf{SEE} \nvdash \varphi$. It follows from the Soundness Theorem (Theorem 4.3)[7] that $\mathsf{SEE} \nvdash \bot$ and hence that $\{\neg\varphi\}$ is consistent and so may be extended to a maximal consistent set T^φ. Define the evidence labeling $\mathcal{A}^\varphi$ by setting

$$\mathcal{A}^\varphi := \{(t,\psi) \in \mathcal{T} \times \mathfrak{L}(\mathsf{SEE}) \mid t : \psi \in T^\varphi\}$$

and define the valuation V^φ by setting

$$V^\varphi := \{p_k \in \mathfrak{L}(\mathsf{SEE}) \mid k \in \mathbb{N} \text{ and } p_k \in T^\varphi\} \ .$$

We wish to show that the evidenced valuation $(V^\varphi, \mathcal{A}^\varphi)$ is an S-model. Definition 3.4 implies that it suffices for us to show that $\mathcal{A}^\varphi$ is an S-evidence function. By Definition 3.1, to say that $\mathcal{A}^\varphi$ is an S-evidence function means that $\mathcal{A}^\varphi$ satisfies each of Constant Specification S, Application, Sum, Checker, and Update. So let us check that $\mathcal{A}^\varphi$ indeed satisfies each of these properties.

[7] In particular, in the proof of Lemma 4.7, we pointed out that $(\mathcal{T} \times \mathfrak{L}(\mathsf{SEE}), \emptyset)$ is an S-model. Therefore, were it the case that $\mathsf{SEE} \vdash \bot$, it would follow by soundness (Theorem 4.3) that $\mathcal{T} \times \mathfrak{L}(\mathsf{SEE}), \emptyset \models \bot$. Since the definition of truth (Definition 3.5) says that the latter is impossible, we conclude that $\mathsf{SEE} \nvdash \bot$.

- $\mathcal{A}^\varphi$ satisfies Constant Specification S.

  Choose $\psi \in S$. It follows by Rule CN and the maximal consistency of T^φ that $(c_k : \psi) \in T^\varphi$. Applying the definition of $\mathcal{A}^\varphi$, we have that $(c_k, \psi) \in \mathcal{A}^\varphi$. It follows that $\mathcal{A}^\varphi$ satisfies Constant Specification S.

- $\mathcal{A}^\varphi$ satisfies Application.

  Assume $(t, \psi \supset \chi) \in \mathcal{A}^\varphi$ and $(s, \psi) \in \mathcal{A}^\varphi$. Applying the definition of $\mathcal{A}^\varphi$, we have that $\big(t : (\psi \supset \chi)\big) \in T^\varphi$ and $(s : \psi) \in T^\varphi$. But then it follows by Axiom E1 and the maximal consistency of T^φ that $\big((t \cdot_\psi s) : \chi\big) \in T^\varphi$. Applying the definition of $\mathcal{A}^\varphi$, we have that $(t \cdot_\psi s, \chi) \in \mathcal{A}^\varphi$. It follows that $\mathcal{A}^\varphi$ satisfies Application.

- $\mathcal{A}^\varphi$ satisfies Sum, Checker, and Update by arguments similar to that for Application (though we use Axioms E2, E3, and E4, respectively).

Conclusion: $\mathcal{A}^\varphi$ is an S-evidence function, and $(\mathcal{A}^\varphi, V)$ is an S-model.

We now wish to prove a property of $(\mathcal{A}^\varphi, V^\varphi)$ called the *Truth Lemma*: $\theta \in T^\varphi$ if and only if $\mathcal{A}^\varphi, V^\varphi \models \theta$. We first prove the Truth Lemma for $\mathfrak{L}(\mathsf{SEE})$-formulas θ with $d(\theta) = 0$ (Definition 4.4) by an induction on the construction of θ.

- Base Case: $\theta = q$, where $q \in \{p_k, \bot, \top\}$.

  If $\theta \in \{\bot, \top\}$, then the result follows by the maximal consistency of T^φ and the definition of truth (Definition 3.5). If $\theta = p_k$, then the result follows by the definition of V^φ and the definition of truth.

- Induction Case: $\theta = (\psi \star \chi)$, where $d(\psi) = d(\chi) = 0$.

  This case follows easily from the induction hypothesis.

- Induction Case: $\theta = \neg\psi$, where $d(\psi) = 0$.

  This case also follows easily from the induction hypothesis.

- Induction Case: $\theta = (t : \psi)$.

  We have $(t : \psi) \in T^\varphi$ if and only if $(t, \psi) \in \mathcal{A}^\varphi$. But the latter is what it means to have that $\mathcal{A}^\varphi, V^\varphi \models t : \psi$.

- Induction Case: $\theta = (t :^{j,\chi} \psi)$.

  By an easy inductive argument (with the induction on the construction of t), we see that $\mathsf{E}(j, \chi) \vdash t : \psi$ or $\mathsf{E}(j, \chi) \nvdash t : \psi$. (This argument really is easy: just look at the axiomatics of $\mathsf{E}(j, \chi)$ in Figure 1.) Applying Lemma 4.7, we have that $\mathsf{SEE} \vdash t :^{j,\chi} \psi$ or $\mathsf{SEE} \nvdash t :^{j,\chi} \psi$. It therefore follows by the maximal consistency of T^φ that $(t :^{j,\chi} \psi) \in T^\varphi$ if and only if $\mathsf{SEE} \vdash t :^{j,\chi} \psi$. But the latter is equivalent to $\mathsf{E}(j, \chi) \vdash t : \psi$ by Lemma 4.7. Since $\mathsf{E}(j, \chi) \vdash t : \psi$ is equivalent to $\mathcal{A}^\varphi, V^\varphi \models t :^{j,\chi} \psi$ by the definition of truth, the result follows.

This completes our argument that the Truth Lemma holds of $\mathfrak{L}(\mathsf{SEE})$-formulas θ with $d(\theta) = 0$. Let us now argue that the Truth Lemma holds for all $\mathfrak{L}(\mathsf{SEE})$-formulas θ; that is, we show that $\theta \in T^\varphi$ if and only if $\mathcal{A}^\varphi, V^\varphi \models \theta$.

It follows from the Reduction Lemma (Lemma 4.6) and the maximal consistency of T^φ that $\theta \in T^\varphi$ if and only if $\theta^\circ \in T^\varphi$. But $d(\theta^\circ) = 0$ by the Reduction Lemma (Lemma 4.6), and so it follows by what we showed above that $\theta^\circ \in T^\varphi$ if and only if $\mathcal{A}^\varphi, V^\varphi \models \theta^\circ$. But since we showed that $(\mathcal{A}^\varphi, V^\varphi)$ is an S-model, it follows from the Reduction Lemma (Lemma 4.6) and soundness (Theorem 4.3) that $\mathcal{A}^\varphi, V^\varphi \models \theta^\circ$ is equivalent to $\mathcal{A}^\varphi, V^\varphi \models \theta$. All together, we have shown that $\theta \in T^\varphi$ if and only if $\mathcal{A}^\varphi, V^\varphi \models \theta$, which is the statement of the Truth Lemma.

Having proved the Truth Lemma, we may finish the overall proof. First, since $\neg\varphi \in T^\varphi$, it follows from the Truth Lemma that $\mathcal{A}^\varphi, V^\varphi \models \neg\varphi$. Applying the definition of truth, $\mathcal{A}^\varphi, V^\varphi \not\models \varphi$. Since $(\mathcal{A}^\varphi, V^\varphi)$ is an S-model, we have $S \not\models \varphi$ by the definition of validity (Definition 3.6). Thus we have shown that from the assumption $\mathsf{SEE} \nvdash \varphi$, which we made near the beginning of this proof, we may conclude that $S \not\models \varphi$. The statement of the theorem follows. □

Soundness (Theorem 4.3) and completeness (Theorem 4.8) show that when we take S to be the set of SEE-axioms, thereby equating these axioms with the "basic statements" that are to be justified by a constant, then the set of SEE-theorems is equal to the set of S-valid $\mathfrak{L}(\mathsf{SEE})$-formulas. Accordingly, SEE exactly characterizes the S-valid formulas for the set S of SEE-axioms.

5 The Courtroom Evidence Example Formalized

Our simplistic example of courtroom evidence was described in the introduction of this paper. In this example, the jury begins with two pieces of evidence.

1. $x_1 : O$

   In words: x_1 (the recording) is evidence that the boss ordered his subordinate to falsify the ledgers. (We used O as a mnemonic for "ordered." O is to be understood as an abbreviation for a propositional letter.)

2. $x_2 : (O \supset G)$

   In words: x_2 (the judge's instructions) is evidence that "if the boss ordered his subordinate to falsify the ledgers, then the boss is guilty of fraud." (We used G as a mnemonic for "guilty." G is to be understood as an abbreviation for a propositional letter.)

Using the symbol X to denote the conjunction $(x_1 : O) \wedge \big(x_2 : (O \supset G)\big)$, it follows by Axiom E1 of SEE (Figure 2) that

$$\mathsf{SEE} \vdash X \supset (x_2 \cdot_O x_1) : G \ .$$

In words: "[given assumptions X,] $x_2 \cdot_O x_1$ is evidence that the boss is guilty of fraud." The combined evidence $x_2 \cdot_O x_1$ represents the jury using its evidence x_2 that $O \supset G$ and its evidence x_1 that O to conclude that G using the principle of Modus Ponens:

$$\frac{O \supset G \quad O}{G} \ .$$

This application of Modus Ponens may be read, "from assumptions $O \supset G$ and O, conclude G." Here the subscript O in the evidence $x_2 \cdot_O x_1$ indicates the important role O plays as the antecedent of the implication $O \supset G$ in this application of Modus Ponens.

But now let us examine the effect of the boss' attorney's successful challenge as to the authenticity of evidence x_1 (the recording), in which the boss' attorney presents further evidence that succeeds in convincing the jury that x_1 (the recording) is not authentic and so should be set aside. We may equate this successful challenge with the elimination $(1, O)$ because this is the elimination that will eliminate the evidence assertion $x_1 : O$ in accord with the boss' attorney's successful challenge of the evidence x_1 (that the boss ordered his subordinate to falsify the ledgers). Proceeding, we have the following derivation in $\mathsf{E}(1, O)$ (Figure 1).

1.	$x_1 : O$	Axiom EV
2.	$(x_2 \cdot_O x_1) : G$	by 1, Rule EAR

That is, the elimination $(1, 0)$ has the effect of eliminating the evidence assertions $x_1 : O$ and $(x_2 \cdot_O x_1) : G$. Applying Lemma 4.7, it follows that

$$\mathsf{SEE} \vdash x_1 :^{1,O} O \quad \text{and} \quad \mathsf{SEE} \vdash (x_2 \cdot_O x_1) :^{1,O} G \ .$$

Using Axioms U4 and U3 of SEE (Figure 2), it follows that

$$\mathsf{SEE} \vdash [1, O]\neg(x_1 : O) \quad \text{and} \quad \mathsf{SEE} \vdash [1, O]\neg\big((x_2 \cdot_O x_1) : G\big) \ ;$$

that is, "after elimination $(1, O)$, x_1 is not evidence that the boss ordered his subordinate to falsify the ledgers" and "after elimination $(1, O)$, $x_2 \cdot_O x_1$ is not evidence that the boss is guilty of fraud."

All together, we have shown that

$$\mathsf{SEE} \vdash X \supset (x_2 \cdot_O x_1) : G \quad \text{and} \quad \mathsf{SEE} \vdash X \supset [1, O]\neg\big((x_2 \cdot_O x_1) : G\big) \ .$$

So while the jury could at first combine its evidence x_1 (the recording) with evidence x_2 (the judge's instructions) to produce evidence $x_2 \cdot_O x_1$ that the boss is guilty of fraud, the boss' attorney's successful challenge of the evidence x_1 that O ("the boss ordered his subordinate to falsify the ledgers") eliminates evidence x_1 for O, leaving the jury without the evidence $x_2 \cdot_O x_1$ that the boss is guilty of fraud.

6 Conclusion

We have presented SEE, the Theory of Simple Evidence Elimination, and we showed that it is sound and complete with respect to its intended semantics. Using a simplistic example of courtroom evidence, we showed how SEE can be used to reason about evidence and evidence elimination. In future work, we plan to extend our theory to one that not only allows for evidence elimination but also evidence *introduction*, whereby a piece t of evidence may be introduced for an assertion φ, which has the effect of making the evidence assertion $t:\varphi$ true. Such a joint theory of evidence, evidence elimination, and evidence introduction would provide a much fuller account in Justification Logic of the dynamics of evidence held by a rational individual.

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